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Discontinuous Initial Value Problems and Asymptotic Expansion of Steady-State Solutions

ROBERT M. LEWIS

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Robert M. Lewis

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Abstract

Part I

Discontinuous initial value problems for symmetric hyperbolic linear differential equations

Initial value problems and mixed initial-boundary value problems for a first order system of partial differential equations are considered. The initial data are assumed to have jump discontinuities. An existence and uniqueness theorem for weak solutions of such problems is obtained under conditions which allow the coefficient matrices to have multiple eigenvalues. The proof depends upon an a priori calculation of the discontinuities of the solution and a reduction to known theorems for problems with smooth initial data.

Part II

Asymptotic expansion of steady-state solutions of symmetric hyperbolic linear differential equations

Initial value and mixed problems for the first order system $Lw = g(x)e^{-i\omega t}$ are considered. The solution is a vector function $w(t,x)$, and x denotes a vector of n space variables. The vector $g(x)$ is assumed to have jump discontinuities. Under appropriate conditions it is shown that the solution has the form $w(t,x) = z(t,x) + e^{-i\omega t}v(x)$ where $z(t,x) \to 0$ as $t \to \infty$. A formal procedure is introduced to obtain an expansion in powers of ω^{-1} of the steady-state function, $v(x)$. The results of Part I are applied in a proof of the fact that the formal expansion is an asymptotic expansion of $v(x)$ for $\omega \to \infty$. The procedure and the results are shown to be applicable to Maxwell's equations.

Part I

Discontinuous initial value problems for symmetric hyperbolic linear differential equations

Introduction

Discontinuous solutions of hyperbolic partial differential equations have often been treated in the mathematical literature. The earlier treatments dealt mainly with second order equations (see e.g. [1], [2]). Recently R. Courant and P.D. Lax[3] extended the theory to first order linear hyperbolic systems and proved an existence theorem for discontinuous solutions of initial value problems. In Part I of this paper we generalize the theory of Courant and Lax in two respects. First we consider more general equations. This generalization is not without interest for it permits the inclusion of some important equations of mathematical physics.* Second we consider initial-boundary value problems rather than pure initial value problems.

In Part II our theory will be applied to the study of asymptotic expansions of solutions of certain partial differential equations.

1. Initial value problems for symmetric hyperbolic equations

Let u be a k component vector function of the (n+1) variables x_o, x_1, ..., x_n (u may be thought of as a row or column matrix, according to the context in which it appears). Let A^ν and B denote k x k matrices with real entries. We will sometimes denote x_o by t (since in physical problems it may represent time) and the 'space variables' x_1, ..., x_n by x. The matrices A^ν

---

* That the present theory is directly applicable to Maxwell's equations is demonstrated in Part II.

and B are taken to be smooth* functions of x and t in the half-space $t \geq 0$. We define a first order linear differential operator

$$(1) \qquad Lu = \sum_{\nu=0}^{n} A^{\nu} u_{x_\nu} + Bu, \qquad \left(u_{x_\nu} = \frac{\partial u}{\partial x_\nu}\right).$$

The operator L will be taken to be symmetric hyperbolic, i.e., the matrices A^ν are assumed to be symmetric, and the matrix A^0 to be positive-definite. These definitions are discussed in [4] and [5]. Without loss of generality we shall take $A^0 = I$, where I is the identity matrix. This is possible because the positive definite matrix A^0 may be written as $A^0 = T'T$, where T is a non-singular matrix. Then the transformation of dependent variables $Tu = v$ transforms L into a symmetric hyperbolic operator on v in which the coefficient of v_t is I.

Linear symmetric hyperbolic differential equations have been extensively studied in recent years[4],[5]. One result of this study is the following existence, uniqueness, and differentiability theorem:

> Theorem 1: Let L be a symmetric hyperbolic operator with smooth coefficients, $\phi(t,x)$ a smooth function defined in the half-space $t \geq 0$, and $g(x)$ a smooth initial function defined in the entire x-space. Then the initial value problem
>
> $$Lu = \phi; \qquad u(0,x) = g(x)$$
>
> has a unique smooth solution, $u(t,x)$, defined in the entire half-space. The value of the solution y at a given point depends on

---

[*] Throughout this paper, the term 'smooth' will be used to describe functions which have continuous derivatives of sufficiently high order for the purpose at hand. The precise order can always be obtained by careful examination of the argument, but will usually be ignored.

the values of the coefficients, and of the functions ϕ and g only in a finite region.*

Since it is our object to extend this theorem to discontinuous functions, we must introduce the notion of a 'weak solution'. For this purpose we consider a space $\mathcal{S}$ of smooth testing functions $\Omega(t,x)$ with compact support. More precisely, $\mathcal{S}$ will be the collection of all smooth k component vector functions Ω each of which vanishes identically outside some compact region R which depends upon Ω and lies in the half space $t > 0$. For any u in C^1, the set of functions with continuous first derivatives, and any Ω in $\mathcal{S}$ we have the identity

$$(2) \qquad \iint \Omega Lu \, dxdt = \iint u \, M \Omega \, dxdt.$$

Here M, the adjoint of L, is defined by

$$(3) \qquad M\Omega = -\sum_{\nu=0}^{n} \left(A^{\nu}\Omega\right)_{x_\nu} + B'\Omega .$$

B' denotes the transpose of the matrix B. The integration in (2) extends over the half-space $t \geq 0$. The identity is obtained simply by integration by parts, the boundary terms vanishing because Ω has compact support. If now we have $Lu = 0$, it follows trivially that

$$(4) \qquad \iint u \, M \Omega \, dxdt = 0 \qquad \text{for all } \Omega \text{ in } \mathcal{S}.$$

Conversely if u is in C^1 and satisfies (4) it follows that $Lu = 0$ in the half-space, $t > 0$.

We may, however, consider more general classes of functions than C^1.

---

* This region is the 'backward characteristic cone'. We will not be concerned with its precise delineation.

Accordingly we introduce the class $\mathcal{K}$ of piece-wise smooth functions: For each vector function u in $\mathcal{K}$, defined in the region of interest $\mathcal{D}$ (here $\mathcal{D}$ is the half-space $t \geq 0$), there are a finite number of smooth hypersurfaces which divide $\mathcal{D}$ into subregions $\mathcal{D}_\nu$ such that the first derivatives of u can be made continuous in the closure of each $\mathcal{D}_\nu$ by appropriate definition of the derivative on the boundary.* If a function u in $\mathcal{K}$ satisfies (4) we say that u is a weak solution of the equation $Lu = 0$. If in addition $u(0,x) = g(x)$, we say that u is a weak solution of the initial value problem

$$Lu = 0, \qquad u(0,x) = g(x). \tag{5}$$

We will now analyze the discontinuities and prove the existence of weak solutions of the problem (5) for appropriate discontinuous functions g.

2. The transport equations

Consider a function u which is a solution of (4) and which is smooth everywhere except on a smooth hypersurface C whose equation is $\phi^o(t,x) = 0$. u and all of its derivatives are assumed to have finite jump discontinuities across C. We introduce a regular transformation $y_j = \phi^j(x_o,\dots,x_n)$, $j = 0,1,\dots,n$, which utilizes C as a coordinate surface. The equation of C becomes $y_o = 0$. In the new coordinates we have

$$Lu = \sum_{j=o}^{n} H^j u_{y_j} + Bu . \tag{6}$$

Here we have introduced the matrices H^j defined by

$$H^j = \sum_{\nu=o}^{n} \phi^j_{x_\nu} A^\nu ; \qquad j = 0,1,\dots,n. \tag{7}$$

---

* Then the discontinuities of u and its first derivatives will be jump discontinuities across the hypersurfaces.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ denote regions on either side of C and let $u = u^1 + u^2$ where $u^1 \equiv 0$ in $\mathcal{D}_2$ and $u^2 \equiv 0$ in $\mathcal{D}_1$. By Gauss' theorem

(8) $$\iint -u^1\left(A^\nu \Omega\right)_{x_\nu} dxdt = \iint \Omega\, A^\nu u^1_{x_\nu}\, dxdt - \int_C \left(\Omega A^\nu u^1\right) \alpha \phi^0_{x_\nu}\, dS \; .$$

Here α is a scalar factor so chosen that $(\phi^0_{x_0}, \dots, \phi^0_{x_n})\alpha$ is a unit vector pointing in the required direction. It follows that

(9) $$\iint u^1 M\, \Omega\, dxdt = \iint \Omega\, Lu^1 dxdt - \int_C \sum_{\nu=0}^{n} \left(\Omega\, A^\nu u^1\right) \alpha \phi^0_{x_\nu}\, dS.$$

By adding (9) to the corresponding equation for u^2, we obtain an equation in which the double integrals vanish because of (4) and the fact that $Lu = 0$ everywhere except on C. We are left with

(10) $$\int_C \sum_{\nu=0}^{n} \Omega\, A^\nu [u]\, \alpha \phi^0_{x_\nu}\, dS = \int_C \Omega\, H^0 [u]\, \alpha dS = 0.$$

The symbol $[u] = u^2 - u^1$ denotes the jump in u across C. Since the components of Ω are arbitrary it follows that

(11) $$H^0 [u] = 0.$$

If the vector $[u]$ is not zero, the matrix H^0 must be singular, i.e.,

(12) $$\det \left(\sum_{\nu=0}^{n} \phi^0_{x_\nu} A^\nu \right) = 0.$$

But if ϕ^0 is a solution of (12), the hypersurface $\phi^0 = 0$ is by definition a characteristic hypersurface for the operator L. Thus if u is discontinuous across a hypersurface C, then C must be characteristic. We shall see later that the same conclusion follows if any derivative of u is discontinuous across C.

At this point we impose an important condition on the class of operators L under consideration, the

multiplicity condition: If $P(t,x,z) = \sum_{\nu=1}^{n} z_\nu A^\nu$, where the z_ν are real numbers not all zero, then the multiplicities of the eigenvalues of the matrix P are independent of t, x, and $z = (z_1,\ldots,z_n)$.*

The multiplicity condition assures us that certain eigenvalues $\alpha_1,\ldots,\alpha_q$ of H^o are identically zero on C and the others are non-zero.** Let R^ν denote the associated orthonormal eigenvectors. From (11), $[u]$ is in the null space of H^o, and hence can be expanded in terms of the null eigenvectors (i.e., those associated with zero eigenvalues):

$$[u] = \sum_{m=1}^{q} \sigma_m^o R^m. \tag{13}$$

The σ_m^o are scalars. Since $Lu = 0$ on both sides of $\phi_o = 0$,

$$\sum_{j=o}^{n} H^j \left[u_{y_j}\right] + B[u] = 0. \tag{14}$$

For $j \neq 0$, u_{y_j} is a derivative tangential to C, so $\left[u_{y_j}\right] = [u]_{y_j}$. Thus

$$H^o\left[u_{y_o}\right] + \sum_{j=1}^{n} H^j [u]_{y_j} + B[u] = 0. \tag{15}$$

---

[*] In the work of Courant and Lax[3], the eigenvalues are required to be distinct (i.e., of multiplicity one). For Maxwell's equations P has three eigenvalues, each of multiplicity two, (see Part II).

[**] It follows from the lemma of Appendix 1 that the eigenvalues of H^o are continuous functions of position on C. If now $\alpha_1,\ldots,\alpha_q$ are identically zero in a region $\mathcal{D}_1$ on the hypersurface C, and another set $\beta_1,\ldots,\beta_p$ were identically zero in a neighboring region $\mathcal{D}_2$ on C, then by continuity, both sets would be zero on the common boundary of $\mathcal{D}_1$ and $\mathcal{D}_2$ in violation of the multiplicity condition.

Since $H^oR^\mu = R^\mu H^o = 0$, $\mu = 1,\dots,q$; multiplication of (15) by R^μ yields

$$(16) \qquad \sum_{j=1}^{n} R^\mu H^j [u]_{y_j} + R^\mu B[u] = 0; \qquad \mu = 1,\dots,q.$$

Substituting (13) into (16),

$$(17) \quad \sum_{j=1}^{n}\sum_{m=1}^{q} R^\mu H^j R^m (\sigma_m^o)_{y_j} + \sum_{j=1}^{n}\sum_{m=1}^{q} R^\mu H^j R^m_{y_j} \sigma_m^o + \sum_{m=1}^{q} R^\mu B R^m \sigma_m^o = 0; \; \mu = 1,\dots,q. \, {}^*$$

Equation (17) is a first order linear system of partial differential equations for the scalar coefficients σ_m^o in the expansion of $[u]$. It is called the transport equation of order zero. Before discussing it further, we will derive the corresponding equations for the discontinuities in the derivatives of u. From (6) we have for $r = 1,2,\dots$

$$(18) \quad \frac{\partial^r}{\partial y_o^r} Lu = \sum_{j=o}^{n} H^j u_{y_o^r y_j} + B u_{y_o^r} + r \sum_{j=o}^{n} H^j_{y_o} u_{y_o^{(r-1)} y_j} + N^{(r-1)} u.$$

Here the operator $N^{(r-1)}$ involves derivatives of u of order (r-1) or less, and we have used the abbreviation

$$u_{y_o^r y_j} = \frac{\partial^{(r+1)} u}{\partial y_o^r \partial y_j} \; .$$

Since $Lu \equiv 0$ on both sides of C,

$$(19) \quad H^o \left[u_{y_o^{(r+1)}}\right] + \sum_{j=1}^{n} H^j \left[u_{y_o^r}\right]_{y_j} + \left(B + rH^o_{y_o}\right) \left[u_{y_o^r}\right] + r \sum_{j=1}^{n} H^j_{y_o} \left[u_{y_o^{(r-1)}}\right]_{y_j} + \left[N^{(r-1)} u\right] = 0.$$

If all derivatives of u of order less than (r+1) are continuous across C, then

---

[*] A derivative of an eigenvector appears in (17). The lemma of Appendix 1 shows that the eigenvalues and eigenvectors of H^o are smooth functions.

$H^o\left[u_{y_o^{(r+1)}}\right] = 0$. If now, some (r+1)st. derivative has a non-zero jump, then $\left[u_{y_o^{(r+1)}}\right] \neq 0$* and H^o is singular. Thus we may assert that if u or any of its derivatives has a jump discontinuity across C, then C is a characteristic surface. Suppose now $\left[u_{y_o^s}\right]$ is known for $s = 0,1,\ldots,(r-1)$. Then the jumps in all derivatives of u of order (r-1) or less are known, and hence $\left[N^{(r-1)}u\right]$ is known. By substituting r for (r+1), we can rewrite (19) in the form

$$(20) \qquad H^o\left[u_{y_o^r}\right] + \sum_{j=1}^{n} H^j\left[u_{y_o^{(r-1)}}\right]_{y_j} + \left[P^{(r-1)}u\right] = 0.$$

Here $\left[P^{(r-1)}u\right]$ involves derivatives of u of order (r-1) or less, so it is known. We now expand $\left[u_{y_o^r}\right]$ in terms of all the eigenvectors of H^o:

$$(21) \qquad \left[u_{y_o^r}\right] = \sum_{m=1}^{k} \sigma_m^r R^m .$$

From (20) and (21) we have

$$(22) \qquad H^o\left[u_{y_o^r}\right] = \sum_{m=q+1}^{k} \alpha_m \sigma_m^r R^m = - \sum_{j=1}^{n} H^j\left[u_{y_o^{(r-1)}}\right]_{y_j} - \left[P^{(r-1)}u\right] .$$

Multiplying by R^s,

$$(23) \qquad \sigma_s^r = -\frac{1}{\alpha_s}\left\{\sum_{j=1}^{n} R^s H^j\left[u_{y_o^{(r-1)}}\right]_{y_j} + R^s\left[P^{(r-1)}u\right]\right\}; \quad s = q+1,\ldots,k.$$

Thus (23) gives the coefficients of the non-null eigenvectors in the expansion of $\left[u_{y_o^r}\right]$ in terms of known functions. To obtain the other coefficients, we multiply (19) by a null eigenvector:

---

* Any (r+1)st derivative can be expressed as a linear combination of $u_{y_o^{(r+1)}}$ and tangential derivatives of r^{th} derivatives of u. Since all r^{th} derivatives are continuous, their tangential derivatives are continuous. Hence $u_{y_o^{(r+1)}}$ must be discontinuous.

$$(24)\qquad \sum_{j=1}^{n} R^{\mu}H^{j}\left[u_{y_0^r}\right]_{y_j} + R^{\mu}(B+rH^{o}_{y_0})\left[u_{y_0^r}\right] = R^{\mu}\left[T^{(r-1)}u\right]; \quad \mu = 1,\ldots,q$$

$\left[T^{(r-1)}u\right]$ involves derivatives of u of order less than r, hence by assumption is known. Upon substituting (21) into (24), we obtain

$$(25)\qquad \sum_{j=1}^{n}\sum_{m=1}^{q} R^{\mu}H^{j}R^{m}(\sigma^{r}_{m})_{y_j} + \sum_{j=1}^{n}\sum_{m=1}^{q} R^{\mu}H^{j}R^{m}_{y_j}\sigma^{r}_{m} + \sum_{m=1}^{q} R^{\mu}(B+rH^{o}_{y_0})R^{m}\sigma^{r}_{m} = g^{r}_{\mu};$$

$$\mu = 1,\ldots,q \;.$$

The symbol g^{r}_{μ} is defined by

$$(26)\qquad g^{r}_{\mu} = R^{\mu}\left[T^{(r-1)}u\right] - \sum_{j=1}^{n} R^{\mu}H^{j}\left[\sum_{s=q+1}^{k}\sigma^{r}_{s}R^{s}\right]_{y_j} - R^{\mu}(B+rH^{o}_{y_0})\sum_{s=q+1}^{k}\sigma^{r}_{s}R^{s} \,.$$

The g^{r}_{μ} are known functions, the σ^{r}_{s} being given by (23) in terms of lower order jumps. Let us establish the convention that quantities with negative superscripts are zero; then since $\sigma^{o}_{s} = 0$ for $s > q$, $g^{o}_{\mu} = 0$ and (25) contains (17) as the special case, $r = 0$. We rewrite (25) as

$$(27)\qquad \sum_{j=1}^{n}\sum_{m=1}^{q} R^{\mu}H^{j}R^{m}(\sigma^{r}_{m})_{y_j} + \sum_{m=1}^{q} \tau^{r}_{\mu m}\sigma^{r}_{m} = g^{r}_{\mu}\,; \qquad \begin{array}{l}\mu = 1,\ldots,q\\ r = 0,1,\ldots\end{array}$$

where

$$(28)\qquad \tau^{r}_{\mu m} = \sum_{j=1}^{n} R^{\mu}H^{j}R^{m}_{y_j} + R^{\mu}(B+rH^{o}_{y_0})R^{m}.$$

It can be shown that (27) is for each r a symmetric hyperbolic system. Therefore we could apply Theorem 1 to assert the existence and uniqueness of its solution. However we can improve on this approach: We shall show that (27) reduces to a system of ordinary differential equations.* To see this, we shall prove the following

---

* For this very useful (and far from obvious) observation, we are indebted to P. Ungar.

> **Lemma:** Let $R_o^1, \ldots, R_o^q$ be orthonormal eigenvectors associated with the same eigenvalue λ_o of the matrix $P = \sum_{\nu=1}^{n} z_\nu A^\nu$, ($z_\nu$ real). Then for each j,

$$R_o^\mu A^j R_o^m = c_j \delta_{\mu m} \tag{29}$$

> (i.e., $R_o^\mu A^j R_o^m = 0$ if $\mu \neq m$, and is independent of μ if $\mu = m$.)

Proof: For any j let $M(\zeta) = P + \zeta A^j$. Let $R^1(\zeta), \ldots, R^q(\zeta)$ be orthonormal eigenvectors of M corresponding to the eigenvalue $\lambda(\zeta)$ where $\lambda(0) = \lambda_o$ and $R^\nu(0) = R_o^\nu$. From the multiplicity condition, it follows by means of the lemma of Appendix 1 that the $R^\nu(\zeta)$ and $\lambda(\zeta)$ are analytic functions in a neighborhood of $\zeta = 0$.

Let R_1^m and λ_1 denote the derivatives with respect to ζ evaluated at $\zeta = 0$ of $R^m(\zeta)$ and $\lambda(\zeta)$ respectively. Differentiating the equation $MR^m = \lambda R^m$ we obtain

$$A^j R_o^m + PR_1^m = \lambda_1 R_o^m + \lambda_o R_1^m ,$$

and multiplying on the left by R_o^μ,

$$R_o^\mu A^j R_o^m = R_o^\mu(\lambda_o - P)R_1^m + \lambda_1 R_o^\mu R_o^m = \lambda_1 \delta_{\mu m} .$$

This completes the proof of the lemma.

Since the matrices H^j are just linear combinations of the matrices A^j it follows from the lemma that

$$R^\mu H^j R^m = 0 \text{ if } \mu \neq m; \quad R^\mu H^j R^\mu = R^1 H^j R^1; \quad \text{for } \mu, m = 1, \ldots, q. \tag{30}$$

Thus (27) may be rewritten

$$(31)\qquad \sum_{j=1}^{n} R^1 H^j R^1 (\sigma^r_\mu)_{y_j} + \sum_{m=1}^{q} \tau^r_{\mu m}\, \sigma^r_m = g^r_\mu\,; \qquad \mu = 1,\dots,q \quad r = 0,1,\dots \; .$$

Expanding H^j as in (7) yields

$$(32)\qquad \sum_{j=1}^{n} R^1 H^j R^1 (\sigma^r_\mu)_{y_j} = \sum_{j=1}^{n} \sum_{\nu=0}^{n} \phi^j_{x_\nu}\, R^1 A^\nu R^1 (\sigma^r_\mu)_{y_j}$$

$$= \sum_{\nu=0}^{n} R^1 A^\nu R^1 \sum_{j=1}^{n} (\sigma^r_\mu)_{y_j} \frac{\partial y_j}{\partial x_\nu} = \sum_{\nu=0}^{n} R^1 A^\nu R^1 (\sigma^r_\mu)_{x_\nu} \; .$$

Here we have used the fact that $(\sigma^r_\mu)_{y_0} = 0$. Thus (31) becomes

$$(33)\qquad \sum_{\nu=0}^{n} R^1 A^\nu R^1 (\sigma^r_\mu)_{x_\nu} + \sum_{m=1}^{q} \tau^r_{\mu m}\, \sigma^r_m = g^r_\mu\,; \qquad \mu = 1,\dots,q \quad r = 0,1,\dots \; .$$

We now consider a family of curves in the tx-space defined parametrically by $x_\nu = x_\nu(s)$; $\nu = 0,1,\dots,n$; where

$$(34)\qquad \dot{x}_\nu = \frac{dx_\nu}{ds} = R^1 A^\nu R^1; \qquad \nu = 0,1,\dots,n.$$

Since $\dot{x}_0 = \dot{t} = 1$, we may identify t with s (up to an additive constant on each curve). Then (33) may be interpreted as ordinary differential equations along these curves, for

$$\sum_{\nu=0}^{n} R^1 A^\nu R^1 (\sigma^r_\mu)_{x_\nu} = \sum_{\nu=1}^{n} \dot{x}_\nu (\sigma^r_\mu)_{x_\nu} = \dot{\sigma}^r_\mu \; .$$

Hence

$$(35)\qquad \dot{\sigma}^r_\mu + \sum_{m=1}^{q} \tau^r_{\mu m}\, \sigma^r_m = g^r_\mu\,; \qquad \mu = 1,\dots,q \quad r = 0,1,\dots \; .$$

We observe that $\sum_{\nu=0}^{n} \dot{x}_\nu \phi^0_{x_\nu} = R^1 H^0 R^1 = 0$ on $\phi^0 = 0$. Hence the curves which we have introduced are tangential to the characteristic surface $\phi^0 = 0$. It can be shown that these curves, which we shall refer to as rays, are identical with the characteristic curves of the single first order partial differential equation (12) for ϕ^0. As such, they are sometimes referred to as the bicharacteristics of the operator L. From the theory of first order partial differential equations, we know that the characteristic surface $\phi^0 = 0$ is generated by an (n-1) parameter family of rays.

Thus for each fixed r, (35) is a system of q linear ordinary differential equations along the rays. Examination of the foregoing argument shows that the coefficients $\tau^r_{\mu m}$ and g^r_μ will be smooth functions provided the hypersurface C is smooth and the σ^ν_m are smooth for $\nu < r$. Given appropriate initial values (these will be derived in the next section), we may then assert that there exists a unique smooth solution of (35). Thus the system of differential equations (35) and algebraic equations (23) can, in principle, be solved recursively for $r = 0, 1, \ldots, r_1$; and for r_1 as large as we wish. These equations will be referred to as the transport equations (of order r).

Remark: The higher transport equations discussed above determine the discontinuities in derivatives of u normal to the characteristic surface, C. In the special case that C never has a tangent plane parallel to the t-axis, we could instead consider derivatives in the t-direction. If, in addition, the coefficients A^ν, B are independent of t this leads to a considerable simplification of the equations. A simplified analysis based on these assumptions is carried out in Part II, Section 6.

3. Initial conditions for the transport equations

In the previous section we were concerned with a single characteristic surface, C, and its associated transport equations. Since, in what follows, we shall have to be concerned with several distinct characteristic surfaces $C^{(p)}$, we will use an additional index (p) on various quantities.

Let Γ be a given smooth (n-1) dimensional manifold in the initial hyperplane, $t = 0$. Let $g(x)$ be smooth on both sides of Γ, while g and its derivatives suffer jump discontinuities across Γ. Let $\gamma_1, \ldots, \gamma_n$ be components of the unit normal to Γ, and let $\phi^{(p)} = \phi^{o(p)}(t,x) = 0$ be the equation of a characteristic surface $C^{(p)}$ whose intersection with the hyperplane $t = 0$ is Γ. Then $\phi^{(p)}_{x_\nu} = \rho\gamma_\nu$ $(\nu = 1, \ldots, n)$ on Γ, where ρ is a proportionality factor.* On Γ, $\phi^{(p)}_{x_o} = \phi^{(p)}_t$ is a solution of

$$(36) \qquad \det\left(\sum_{\nu=o}^{n} \phi^{(p)}_{x_\nu} A^\nu\right) = \det H^{o(p)} = \det\left(\phi^{(p)}_t + \rho E\right) = 0, \text{ on } \Gamma,$$

where

$$(37) \qquad E = \sum_{\nu=1}^{n} \gamma_\nu A^\nu .$$

Let R^j; $j = 1, \ldots, k$ denote orthonormal eigenvectors of E (hence of $H^{o(p)}$ on Γ) and λ_j the corresponding eigenvalues. Then the eigenvalues of $H^{o(p)}$ are $\phi^{(p)}_t + \rho\lambda_j$. By virtue of (36), for each p and for some j,

---

[*] Geometrically, this is the condition that the projection onto the hyperplane $t = 0$ of a vector normal to $C^{(p)}$ at a point on Γ shall be parallel to the normal vector $(\gamma_1, \ldots, \gamma_n)$ of Γ. The condition is equivalent to the so-called 'strip manifold condition' in the theory of first order partial differential equations. See, e.g., [1], Chapter II.

(38) $$\phi_t^{(p)} + \rho\lambda_j = 0.$$

There will be as many distinct characteristic surfaces, $C^{(p)}$, intersecting Γ as there are distinct solutions $\phi_t^{(p)}$ of (38), hence as many as there are distinct eigenvalues of E. The multiplicity condition assures us that the number of these distinct characteristic surfaces $C^{(p)}$ is the same at every point of Γ. On each such $C^{(p)}$ a different set of eigenvectors is null for $H^{o(p)}$ (those R^j for which λ_j satisfies (38)), but the totality of null eigenvectors for all the surfaces $C^{(p)}$ are the totality of the eigenvectors of E.

Let $[\ ]^p$, $[\ ]^{\Gamma}$ denote jumps across $C^{(p)}$ and Γ respectively. Since the eigenvectors R^j form a complete set the jump in initial values of u across Γ has the unique decomposition

(39) $$[u]^{\Gamma} = [g]^{\Gamma} = \sum_{m=1}^{k} \sigma^o_{om} R^m .$$

We rewrite (13), including the additional index, p:

(40) $$[u]^p = \sum_{m=1}^{k} \sigma^o_{pm} R^m .$$

Here $\sigma^o_{pm} = 0$ if R^m is not a null eigenvector of $H^{o(p)}$. The initial conditions for the σ^o_{pm} are obtained from the fact that

(41) $$\sum_p [u]^p = [u]^{\Gamma}, \qquad \text{on } \Gamma.$$

Multiplying by any eigenvector R^v,

(42) $$\sum_p \sigma^o_{pv} = \sigma^o_{ov}, \qquad \text{on } \Gamma.$$

But for every v, R^v is a null eigenvector of H^o on exactly one of the characteristic surfaces, say $C^{(q)}$. Hence $\sigma^o_{pv} = 0$ if $p \neq q$ and (42) becomes

$$\sigma^o_{qv} = \sigma^o_{ov}\,, \qquad \text{on } \Gamma. \tag{43}$$

Thus (43) determines the initial values of all the σ^o_{pm}.

To obtain the initial conditions for the higher transport equations, we suppose now that $\left[u_{y_o^\rho}\right]^p$ have been determined on Γ for all p and for $\rho = 0,1,\ldots,r-1$. Then the jumps in all derivatives of order less than r can be calculated. (Note that we have a different variable y_o for each of the $C^{(p)}$ and the derivative $\frac{\partial}{\partial y_o}$ is directed normally to $C^{(p)}$.) For each characteristic surface $C^{(p)}$, it is convenient to introduce two directions of differentiation at each point of Γ: $\frac{\partial}{\partial N}$ (normal to Γ in the plane $t = 0$), and $\frac{\partial}{\partial s}$ (normal to Γ and tangential to $C^{(p)}$). The corresponding directions are indicated in Fig. 1. Since the three directions (y_o, N, s) are all normal to Γ they are linearly dependent, hence we may expand u_{N^r} as

$$u_{N^r} = \sum_{\nu=0}^{r} e^p_\nu \, u_{y_o^{(r-\nu)} s^\nu} \;. \tag{44}$$

Here the e^p_ν are known angle factors, and $e^p_o \neq 0$ since the plane $t = 0$ is non-characteristic.* From (44)

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* A characteristic surface can never have a tangent plane parallel to the plane $t = 0$. This follows from (12).

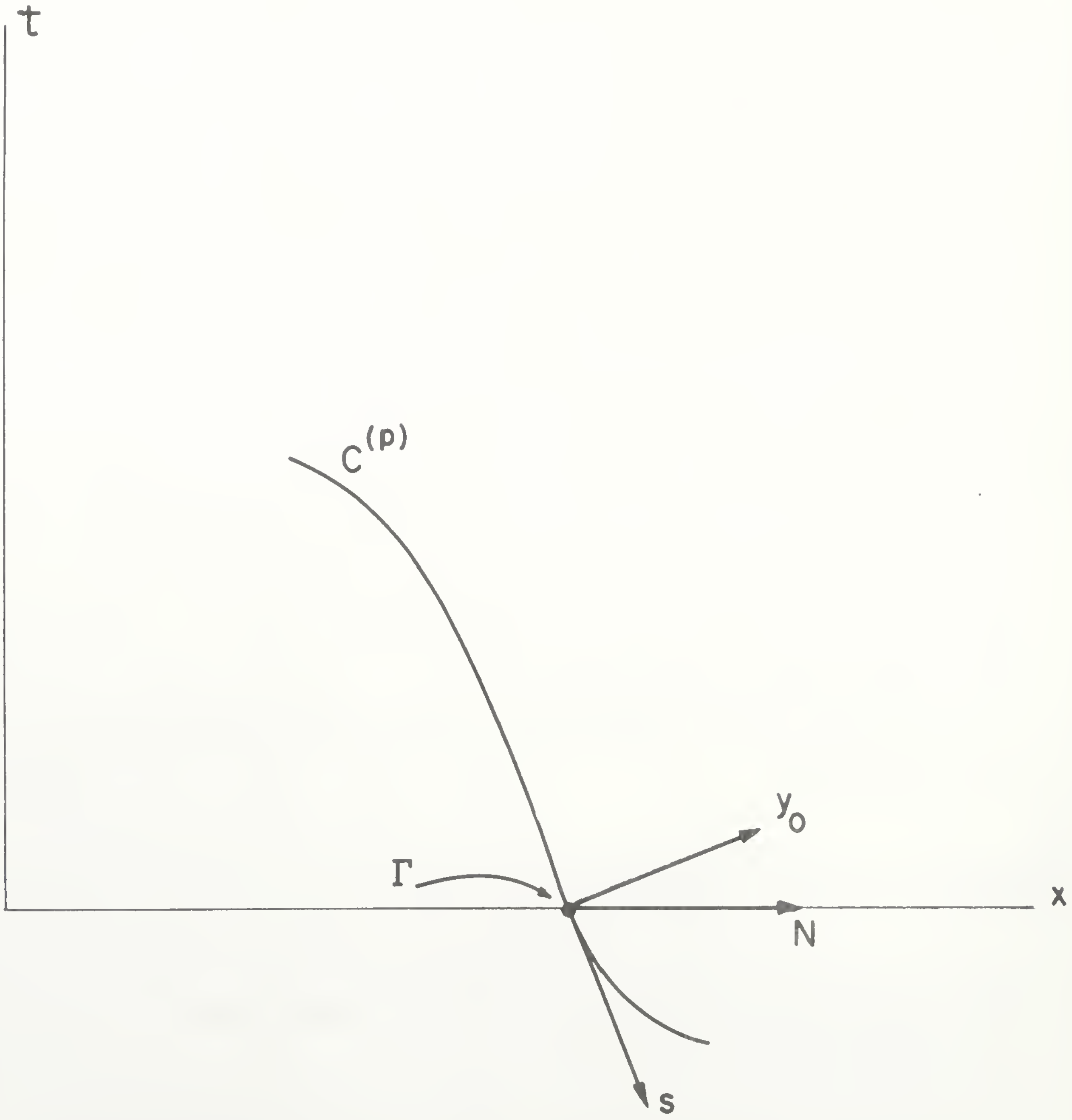

Figure 1

Relative orientations of the directions y_o, N, s on the initial plane.

(45) $$\left[u_{N^r}\right]^p = e_0^p \left[u_{y_0^r}\right]^p + \sum_{\nu=1}^{r} e_\nu^p \left[u_{y_0^{(r-\nu)}}\right]_{s^\nu}^p .$$

By (45), $\left[u_{N^r}\right]^p$ and $\left[u_{y_0^r}\right]^p$ determine each other uniquely. As before, we rewrite (21) as

(46) $$\left[u_{y_0^r}\right]^p = \sum_{m=1}^{k} \sigma_{pm}^r \; R^m .$$

Let

(47) $$\left[u_{N^r}\right]^p = \sum_{m=1}^{k} \beta_{pm}^r \, R^m .$$

Then from (45)

(48) $$\beta_{p\mu}^r = R^\mu \left[u_{N^r}\right]^p = e_0^p \; \sigma_{p\mu}^r + \gamma_{p\mu}^r$$

where

(49) $$\gamma_{p\mu}^r = R^\mu \sum_{\nu=1}^{r} e_\nu^p \left[u_{y_0^{(r-\nu)}}\right]_{s^\nu}^p .$$

By (48), the $\beta_{p\mu}^r$ and $\sigma_{p\mu}^r$ determine each other uniquely in terms of the $\gamma_{p\mu}^r$ which we may assume to be known. The initial conditions for the $\sigma_{p\mu}^r$ are obtained from the fact that

(50) $$\sum_p \left[u_{N^r}\right]^p = \left[u_{N^r}\right]^\Gamma = \left[g_{N^r}\right]^\Gamma, \qquad \text{on } \Gamma.$$

As before, we have

(51) $$\left[u_{N^r}\right]^\Gamma = \left[g_{N^r}\right]^\Gamma = \sum_{m=1}^{k} \sigma_{om}^r \, R^m .$$

Multiplying by any R^v, (50) becomes

$$\sigma^r_{ov} = \sum_p \beta^r_{pv} = \sum_p \left(e^p_o\,\sigma^r_{pv} + \gamma^r_{pv}\right), \qquad \text{on } \Gamma. \tag{52}$$

The σ^r_{pv} are predetermined by (23) except when R^v is a null eigenvector of $H^{o(p)}$. This occurs on exactly one of the characteristic surfaces, say $C^{(q)}$. Thus σ^r_{qv} is not determined by (23) and its initial values are given by

$$e^q_o\,\sigma^r_{qv} + \gamma^r_{qv} = \sigma^r_{ov} - \sum_{p\neq q}\left(e^p_o\,\sigma^r_{pv} + \gamma^r_{pv}\right), \text{ on } \Gamma. \tag{53}$$

The initial values of the σ^r_{qv} will be smooth functions along Γ provided the σ^r_{ov} are smooth, i.e., provided the $\left[g_{N^r}\right]^{\Gamma}$ are smooth. But this is guaranteed by our assumption that g is smooth on both sides of Γ. It follows that the solutions of the transport equations will be smooth functions of the parameters along Γ, hence smooth functions of position on the characteristic surface, (i.e., not only in the ray direction).

4. Existence and uniqueness theorem for the discontinuous initial value problem

For the proof of the theorem which we are about to present, it is necessary that we be able to solve the transport equations (plus initial conditions) along the characteristic surfaces $C^{(p)}$ emanating from the manifold Γ. It is clear from the foregoing that this can be done in the region $0 \leq t < t_1$, where t_1 is the smallest value of t for which one of the hypersurfaces $C^{(p)}$ has a singularity. Singularities of characteristic surfaces bearing non-zero jumps will be referred to as caustics. Whether or not caustics appear in a given initial value problem can not be determined solely by examining the solutions ϕ of the single first order partial differential equation (12) whose solution surfaces $\phi = 0$ contain Γ; for it is possible that the function $g(x)$ may be such that although the characteristic surface $C^{(p)}$ emanating from Γ has a singularity, that surface does not play a role, the corresponding initial values for the transport equations being all zero. It is convenient to formulate the following

caustic condition: The problem is such that the characteristic surfaces across which u and its derivatives have non-zero jumps are free of caustics.

We may now state

Theorem 2: Given the initial value problem

(54) $$Lu = 0$$

(55) $$u(0,x) = g(x).$$

Here L is a symmetric hyperbolic operator with smooth coefficients which satisfies the multiplicity condition and $g(x)$ is smooth on either side of a smooth $(n-1)$ dimensional manifold Γ in the plane $t = 0$, while g and its derivatives have jump discontinuities across Γ. There exists one and only one weak solution u in the class $\mathcal{K}$, for t in the interval $0 \leq t \leq t_1$.* u is smooth for all x and t in that interval except on the characteristic surfaces emanating from Γ. Across these surfaces u and its derivatives have jump discontinuities which satisfy the transport equations. If the caustic condition is satisfied we may take $t_1 = +\infty$.

Proof: Let $v(t,x)$ be an auxiliary function** which is smooth everywhere in the region $0 \leq t < t_1$ except on the characteristic surfaces $C^{(p)}$ emanating from Γ. v is to be so chosen that its jumps and those of its derivatives across these surfaces are equal to the unique solutions of the transport equations and initial conditions for u. Then the function $\Psi = Lv$ is smooth, for by construction the function Lu and all of its derivatives are continuous across the $C^{(p)}$. Let $u = v + w$ where w is a solution of the inhomogeneous equation

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* t_1 is the smallest value of t for which a caustic appears.

** The device of introducing this function and much of the foregoing construction are due to R. Courant and P. Lax[3].

$$(56) \qquad Lw = -\Psi .$$

The initial conditions on w are

$$(57) \qquad w(0,x) = g(x) - v(0,x).$$

By construction of v, both $\Psi = Lv$ and $w(0,x)$ are smooth. Therefore by Theorem 1, the solution w exists and is smooth. It follows that $u(0,x) = g(x)$ and $Lu = 0$ everywhere except on the surfaces $C^{(p)}$ where u satisfies the transport equations. Now as in (9) and (10),

$$(58) \qquad \iint uM\,\Omega\,dxdt = \iint \Omega\,Lu\,dxdt + \sum_p \int_{C^{(p)}} \Omega\,H^{o(p)}[u]^p \alpha\,dS .$$

But $Lu = 0$, and $H^{o(p)}[u]^p = 0$. Hence (4) is satisfied. Thus it is proved that there exists a weak solution u.

For the uniqueness, let $u = u_1 - u_2$, where u_1 and u_2 are solutions in $\mathcal{K}$. Then u is in $\mathcal{K}$ and is a weak solution of the problem with $g(x) \equiv 0$. The discontinuities of u and its first derivatives can occur only on characteristic surfaces. These always intersect the plane $t = 0$. But the initial values for the transport equations on these surfaces are zero. Hence the discontinuities are zero. It follows that u is continuously differentiable, so u is a strong solution. Therefore, from the uniqueness assertion of Theorem 1, $u \equiv 0$. This proves the uniqueness and completes the proof of the theorem.

5. Initial-boundary value problems for symmetric hyperbolic equations

In addition to initial value problems for symmetric hyperbolic equations, we may also consider the so-called mixed, or initial-boundary value problems. If $\beta(x) = 0$ is the equation of any smooth surface, B, in the x-space, the transformation $z_\nu = \alpha_\nu(x)$; $\nu = 1,\dots,n$; with $\alpha_n = \beta$, maps the surface B into the plane $z_n = 0$, while preserving the symmetric hyperbolic nature of L and the smoothness of its coefficients. Thus without loss of generality we may take our boundary to be $x_n = 0$. On this boundary we will require the solution u to satisfy a certain type of homogeneous boundary condition* which we will now explain.

The vector u is an element of a k-dimensional Euclidean vector space V. Consider a subspace T of V with the following properties

1. The matrix A^n is non-positive over T, i.e., $uA^nu \leq 0$ for every u in T.
2. T is maximal, i.e., the dimension of T is as large as the dimension of any subspace having property 1.

The boundary condition may now be expressed simply as

$$(65) \qquad u(x,t) \text{ in } T, \qquad \text{for} \quad x_n = 0.$$

If $T^\perp$ is the orthogonal complement of T, and $T^\perp$ is spanned by the vectors

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* In Maxwell's equations, the conditions on E and H at the surface of a perfect conductor can be shown to be conditions of this type. See Part II, Section 2.

$b^1,\dots,b^m$, the condition may also be stated as

$$(66) \qquad ub^s = 0; \qquad s = 1,\dots,m, \quad \text{for} \quad x_n = 0.$$

Here ub^s denotes the inner product. The vectors b_s, and the space T need not be fixed but may be functions of the point on the boundary. We will take them to be smooth functions. For problems involving a boundary we will make use of the following generalization of Theorem 1: [6], [7] .

> Theorem 1-B: Let L be a symmetric hyperbolic operator with smooth coefficients, let $\phi(t,x)$ be a smooth function defined in the domain $\mathcal{D}$: $t \geq 0$, $x_n \geq 0$; and $g(x)$ be a smooth function defined in the domain $x_n \geq 0$ which satisfies the boundary condition: g in T, for $x_n = 0$ and $t = 0$. Then the initial-boundary value problem
>
> $$Lu = \phi; \quad u(0,x) = g(x); \quad u \text{ in } T, \quad \text{for } x_n = 0$$
>
> has a unique smooth solution $u(t,x)$ defined in $\mathcal{D}$. The value of the solution u at a given point depends on the values of the coefficients, the functions ϕ and g, and the boundary conditions only in a finite region.

Weak solutions will be defined as before, the only additional requirements being the boundary condition (65) and the requirement that the region R which is the support of the testing function Ω shall not intersect the half-space $x_n \leq 0$.

The entire discussion in Sections 1, 2, 3 now remains valid in the region $0 \leq t < \min(t_1,t_2)$ where t_2 is the smallest value of t for which one of the characteristic surfaces emanating from Γ intersects the boundary. In order to extend our existence theorem beyond t_2 we must first extend our a priori calculation of the discontinuities of u. This is done in the next section.

6. Initial conditions for the transport equations at a boundary

Suppose that a characteristic surface C emanating from the manifold Γ in the initial hyperplane intersects the boundary B in an (n-1) dimensional manifold Δ. Δ will be smooth in the region $t < t_1$, for there C is smooth. We now consider all the characteristic surfaces which intersect B in Δ and attempt to choose initial conditions for the transport equations on these surfaces in such a way that the jumps in u and its derivatives will be compatible with the boundary conditions (65).

Let $\delta_0, \dots, \delta_{n-1}, 0$ denote components of the unit normal to Δ in the plane $x_n = 0$. Let $\phi^{o(p)} = 0$ be the equation of a characteristic surface $C^{(p)}$ whose intersection with B is Δ. Then $\phi^{o(p)}_{x_\nu} = \rho\delta_\nu$, on Δ $(\nu = 0, \dots, n-1)$ where ρ is a proportionality factor. Here $\phi^{o(p)}_{x_n}$ is a solution of

$$\det H^{o(p)} = \det\left(\sum_{\nu=0}^{n} \phi^{o(p)}_{x_\nu} A^\nu\right) = \det\left(\phi^{o(p)}_{x_n} A^n + \rho F\right) = 0; \text{ on } \Delta \tag{67}$$

where

$$F = \sum_{\nu=0}^{n-1} \delta_\nu A^\nu. \tag{68}$$

Equation (67) will, in general, have complex roots $\phi_{x_n}^{o(p)}$. For each distinct real root there will be a corresponding characteristic surface through Δ. We assume that the manifold Γ is bounded away from the boundary, B. Then it is easily seen that there will be a point P on Δ of minimum t, and that at P, $\delta_1 = \dots = \delta_{n-1} = 0$. Hence, at P, $\delta_0 = 1$ and $F = I$. It follows from the continuity of F; that in some neighborhood of P, F is non-singular. We now impose a new condition on the problem; the

shadow condition: F shall be non-singular throughout $\triangle$. This condition as suggested by its name has an interesting geometric interpretation which will be explained below. It follows from the shadow condition that F is positive definite on $\triangle$. Therefore we may set $F = JJ'$ where J is non-singular and has an inverse S. Let $G = SA^nS'$. Then G is symmetric and we may write

$$(69) \qquad (G-\alpha_p)\, Q^{(p)} = 0; \qquad p = 1,\dots,k.$$

Here the $Q^{(p)}$ are a complete set of orthogonal eigenvectors of G, which we may normalize by the condition

$$(70) \qquad Q^{(p)}SS'Q^{(p)} = 1; \qquad p = 1,\dots,k.$$

The eigenvalues α_p are real. Set $R^{(p)} = S'Q^{(p)}$; $p = 1,\dots,k$. It follows from (70) that the $R^{(p)}$ are unit vectors. They are clearly linearly independent. From (69), $J(SA^nS'-\alpha_p)J'R^{(p)} = 0$, or

$$(71) \qquad (A^n-\alpha_pF)R^{(p)} = 0; \qquad p = 1,\dots,k.$$

For each non-zero α_p, set $\phi^{o(p)}_{x_n} = -\rho/\alpha_p$; then

$$(72) \qquad H^{o(p)}R^{(p)} = \left(\phi^{o(p)}_{x_n} A^n + \rho F\right) R^{(p)} = 0; \quad \text{if } \alpha_p \neq 0.$$

Equation (72) expresses the fact that for each distinct non-zero α_p, there is a distinct characteristic surface, $C^{(p)}$, passing through $\triangle$, and $R^{(p)}$ is a null eigenvector of the associated matrix $H^{o(p)}$. Since we may have multiple roots α_p, we may have several distinct $R^{(p)}$ associated with the same matrix $H^{o(p)}$. As we have seen earlier, the jump in u across $C^{(p)}$, $[u]^p$, can be expressed in terms of the associated null eigenvectors. Therefore

$$[u]^{\triangle} = \sum_{q} [u]^{q} = \sum_{p=1}^{k} \sigma_p^o R^{(p)} . \tag{73}$$

Since we have used only the non-zero α_p, we may assume that $\sigma_p^o = 0$ in (73) if $\alpha_p = 0$. (If A^n is singular there will be zero eigenvalues α_p. For these α_p we have solutions of (72) with $\rho = 0$. In this case the boundary is itself a characteristic surface.) We apply the boundary condition in the form (66):

$$\left[b^{\nu}u\right]^{\triangle} = b^{\nu}[u]^{\triangle} = \sum_{p=1}^{k} b^{\nu}R^{(p)}\sigma_p^o = 0; \qquad \nu = 1,\ldots,m . \tag{74}$$

Of course, some of the σ_p^o are already determined (roughly speaking, those associated with a characteristic surface emanating from the initial plane or from the boundary at an earlier time). Our object will be to show that (74) can be solved uniquely for the undetermined σ_p^o in terms of the determined ones. It is easy to see that $\sigma_p^o \left\{\begin{matrix}\text{will} \\ \text{will not}\end{matrix}\right\}$ have a preassigned value at a point on $\triangle$ if the ray at that point (oriented in the direction of increasing t) proceeds $\left\{\begin{matrix}\text{out of} \\ \text{into}\end{matrix}\right\}$ the region $x_n > 0$, i.e., if $\dot{x}_n = R^{(p)}A^nR^{(p)}$ is $\left\{\begin{matrix}< 0 \\ > 0\end{matrix}\right\}$. (We shall see that the possibility $\dot{x}_n = 0$ is precluded by the shadow condition.)

We digress to explain the significance of the shadow condition: Since the $R^{(p)}$ are linearly independent they may be made the columns of a non-singular matrix R. Then the matrix $R'FR = M = (m_{pq})$ has entries $m_{pq} = R^{(p)}FR^{(q)}$. But $R^{(p)}FR^{(q)} = R^{(p)}TT'R^{(q)} = Q^{(p)}Q^{(q)} = 0$ if $p \neq q$. Hence M is a diagonal matrix and $\det M = \prod_{p=1}^{k} R^{(p)}FR^{(p)}$. The shadow condition now guarantees that

$$R^{(p)}FR^{(p)} \neq 0; \qquad p = 1,\ldots,k . \tag{75}$$

From (71),

$$R^{(q)}A^nR^{(p)} = \alpha_p R^{(q)}FR^{(p)}. \tag{76}$$

For all characteristic surfaces $C^{(p)}$ except the boundary itself $\alpha_p \neq 0$, hence $\dot{x}_n = R^{(p)}A^n R^{(p)} \neq 0$. But $\dot{x}_n = 0$ if and only if the ray is tangent to the boundary. Thus the shadow condition suffices to ensure that no ray will be tangent to the boundary. In the language of physical applications, we say that there are no shadows in the problem.

Returning to our examination of the linear algebraic system, (74), let $R^{(p)}A^n R^{(p)}$ be $\left\{\begin{matrix} > 0 \text{ for } p = 1,\ldots,s \\ \leq 0 \text{ for } p = s+1,\ldots,k \end{matrix}\right\}$. We rewrite (74) placing known quantities on the right side

$$\sum_{p=1}^{s} b^\nu R^{(p)} \sigma_p^0 = -\sum_{p=s+1}^{k} b^\nu R^{(p)} \sigma_p^0 ; \quad \nu = 1,\ldots,m . \tag{77}$$

Our object of proving that (77) has a unique solution will be attained if we can show that $s = m$ and the coefficient matrix $(b^\nu R^{(p)})$ is non-singular. To this end, we shall employ some simple lemmas proved in Appendix 2. From (76), $R^{(p)}A^n R^{(q)} = 0$ if $p \neq q$. Let $v = \sum_{p=1}^{s} c_p R^{(p)}$. Then $vA^n v = \sum_{p=1}^{s} c_p^2 R^{(p)}A^n R^{(p)} > 0$ unless $v = 0$. Hence $R^{(1)},\ldots,R^{(s)}$ span a subspace T_1 over which A^n is positive. Similarly $R^{(s+1)},\ldots,R^{(k)}$ span a subspace T_2 over which A^n is non-positive. It follows from lemmas 4 and 2 that $\dim T_1 \leq m$ and $\dim T_2 \leq k-m$ where m is the number of positive eigenvalues of A^n. From lemma 3, m is also the dimension of $T^\perp$ (which is as it is used in (66), (74), and (77)). But $\dim T_1 + \dim T_2 = k$. Hence $\dim T_1 = s = m$. Thus the coefficient matrix $(b^\nu R^{(p)})$ of (77) is square. Suppose it were singular. Then there would exist non-trivial numbers d_p such that $\sum_{p=1}^{m} d_p\, b^\nu R^{(p)} = 0$; $\nu=1,\ldots,m$. Set $u = \sum_{p=1}^{m} d_p R^{(p)}$. Then $ub^\nu = 0$; $\nu=1,\ldots,m$; and $uA^n u = \sum_{p=1}^{m} d_p^2 R^{(p)}A^n R^{(p)} > 0$. But this contradicts the boundary conditions (see Section 5). Hence the coefficient matrix is non-singular and the solution of (77) exists and is unique. That solution; $\sigma_1^0,\ldots, \sigma_m^0$; provides the initial values for the transport equation of order zero.

To obtain the initial conditions for the higher transport equations we proceed as follows: Let s be a unit vector in the ray direction, and $\frac{\partial}{\partial s}$ denote differentiation in that direction. Now s and y_o being orthogonal directions, they determine a two-dimensional plane which must intersect the (n-dimensional) boundary in a direction, N which lies in the boundary, but which cannot lie in $\triangle$. This is true because y_o is orthogonal to $\triangle$ and the shadow condition guarantees that s is not in $\triangle$, (see Fig. 2). We now differentiate the boundary condition (66) in the direction N, r times to obtain

$$(78)\qquad u_{N^r} b^{\nu} + r u_{N^{(r-1)}} b^{\nu}_N + \cdots = 0; \quad \text{on } x_n = 0; \quad \nu = 1,\dots,m;$$

or

$$(79)\qquad \left[u_{N^r}\right]^{\triangle} b^{\nu} = -r\left[u_{N^{(r-1)}}\right]^{\triangle} b^{\nu}_N - \cdots; \quad \text{on } x_n = 0; \quad \nu = 1,\dots,m.$$

As in Section 2, we express u_{N^r} in terms of derivatives in the y_o and s directions:

$$(80)\qquad u_{N^r} = \sum_{\nu=o}^{r} e^p_{\nu}\, u_{y_o^{(r-\nu)} s^{\nu}}$$

or

$$(81)\qquad \left[u_{N^r}\right]^p = e^p_o \left[u_{y_o^r}\right]^p + \sum_{\nu=1}^{r} e^p_{\nu} \left[u_{y_o^{(r-\nu)}}\right]^p_{s^{\nu}} .$$

It is important to note that the shadow condition guarantees that the directions s and N do not coincide, hence $e^p_o \neq 0$. Suppose now that $\left[u_{y_o^{\rho}}\right]^p$ has been determined for each characteristic surface $C^{(p)}$ which intersects the boundary in the manifold $\triangle$, for $\rho = 0,1,\dots,r-1$. Then the jumps in all derivatives of order less than r are known. To avoid

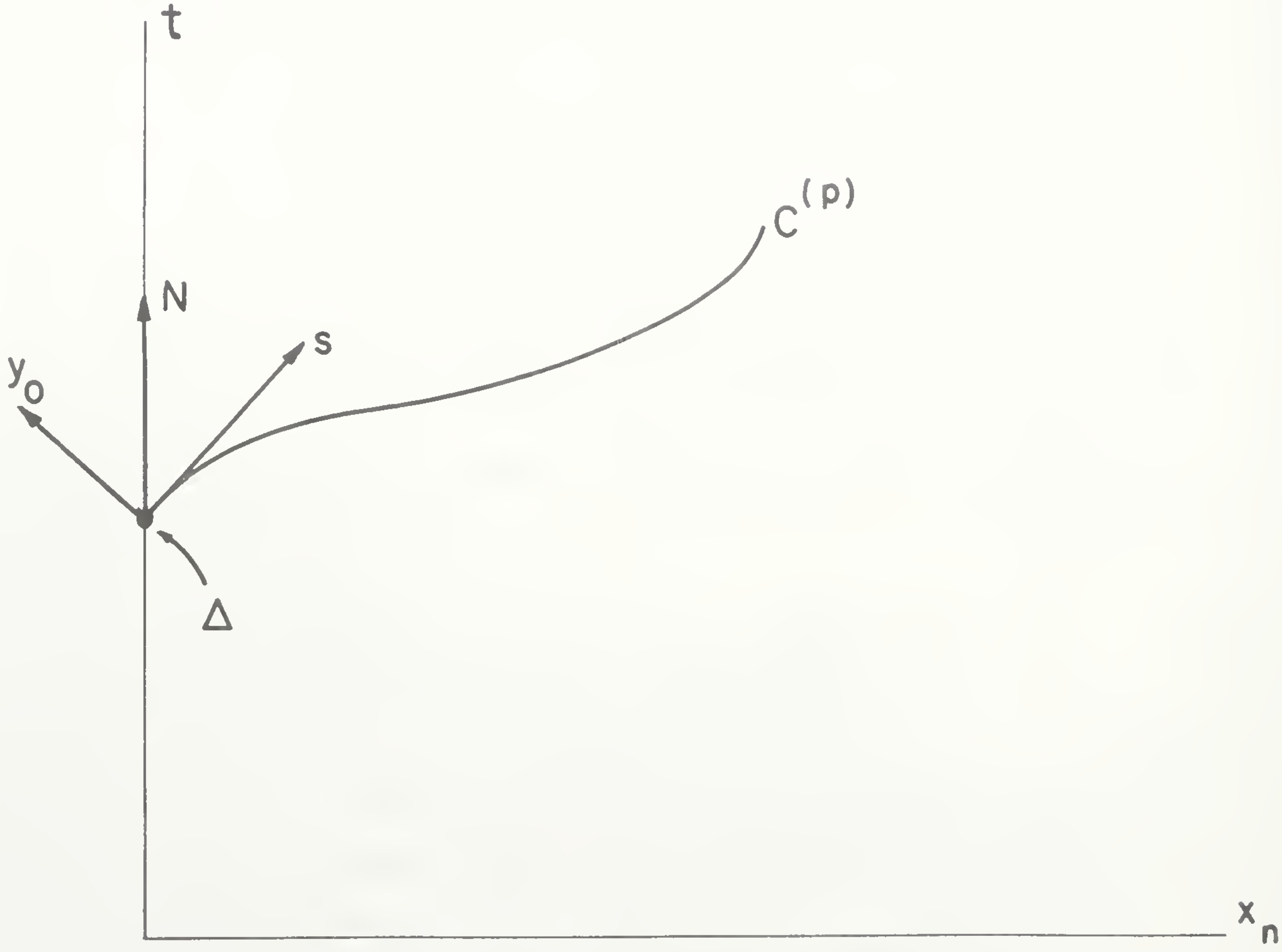

Figure 2

Relative orientations of the directions y_o, N, s at the boundary.

unnecessary writing, let W_j denote known quantities. Then

$$(82) \qquad \left[u_{N^r}\right]^{\triangle} = \sum_p \left[u_{N^r}\right]^p = \sum_p e_o^p \left[u_{y_o^r}\right]^p + W_1 .$$

From (79)

$$(83) \qquad \sum_p e_o^p \left[u_{y_o^r}\right]^p b^\nu = W_2 ; \qquad \nu = 1,\ldots,m.$$

On each $C^{(p)}$, $\left[u_{y_o^r}\right]^p$ can be expressed in terms of the eigenvectors of $H^{o(p)}$, the coefficients of the eigenvectors which are not null being known from (23). The totality (for all $C^{(p)}$) of the null eigenvectors are simply the $R^{(p)}$; $p = 1,\ldots,k$. Furthermore the coefficients of $R^{(m+1)},\ldots,R^{(k)}$ are known; hence

$$(84) \qquad \sum_{p=1}^{m} b^\nu R^{(p)} e_o^p \sigma_p^r = W_3 ; \qquad \nu = 1,\ldots,m .$$

Comparison with (77) now shows that (84) can be solved uniquely for the quantities $e_o^p \sigma_p^r$; $p = 1,\ldots,m$; hence, since $e_o^p \neq 0$, for the σ_p^r, which provide the initial values for the transport equation of order r.

7. Existence and uniqueness theorem for the discontinuous initial boundary value problem

In order to generalize the theorem of Section 3 to problems with boundaries we need only impose upon the auxiliary function v the additional requirement that it satisfy the boundary condition. This can be done because the discontinuities across the characteristic surfaces which intersect the boundary are such as to be compatible with the boundary conditions. Now, by Theorem 1-B we can require that the function w also satisfy the boundary condition. Then the function u = v + w will satisfy the boundary condition.

The uniqueness argument requires an obvious modification. Thus we have

Theorem 2-B: Given the initial boundary value problem

$$Lu = 0 \tag{85}$$

$$u(0,x) = g(x) \tag{86}$$

$$u(x,t) \text{ in } T, \quad \text{for } x_n = 0. \tag{87}$$

Here L is a symmetric hyperbolic operator with smooth coefficients defined in the domain $\mathcal{D}$: $t \geq 0$; $x_n \geq 0$; $g(x)$ is smooth on either side of a smooth (n-1)-dimensional manifold Γ in the plane $t = 0$, while g and its derivatives have jump discontinuities across Γ and g satisfies the boundary condition: g in T for $x_n = 0$ and $t = 0$. Let the multiplicity and shadow conditions be satisfied. Then there exists one and only one solution u in the class $\mathcal{K}$ in the region $0 \leq t < t_1$, $x_n \geq 0$. u is smooth everywhere in that region except on the characteristic surfaces emanating from Γ and from the intersections Δ (which may be multiple)[*] of such surfaces with the boundary, $x_n = 0$. Across these surfaces, u and its derivatives suffer jump discontinuities which satisfy the transport equations. If the caustic condition is satisfied we may take $t_1 = +\infty$.

note: By virtue of the remarks made at the beginning of Section 5 the above theorem is immediately valid for any domain $\mathcal{D}$ which can be mapped by a smooth transformation of the space coordinates onto the domain $t \geq 0$; $x_n \geq 0$*. The theorem is also valid for more general domains such as the interior of a cylinder with generators parallel to the t-axis or domains bounded by several smooth surfaces. In these cases we must employ the analogues of Theorem 1-B valid for such domains [6], [7].

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[*] It is clear that our construction, and hence this theorem are valid for multiple 'reflections' from the boundary.

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[*] It should be noted that the mapping induces a change in the statement of the boundary condition and the shadow condition. Thus, e.g., if $\beta(x) = 0$ defines the boundary, the boundary condition becomes: $u(x,t)$ in T for $\beta(x) = 0$, where T is the maximal subspace such that the matrix $\sum_{\nu=1}^{n} \beta_{x_\nu} A^\nu$ is non-positive over T.

Appendix 1

Lemma: Let P, A be real symmetric matrices of dimension k; let $M(\zeta) = P + \zeta A$ have eigenvalues whose multiplicities are the same for every real value of ζ. Let $\lambda(\zeta)$ be an eigenvalue of M with associated orthonormal eigenvectors $R^1(\zeta)$, $\ldots, R^q(\zeta)$. (Thus q is independent of ζ.) Then $\lambda(\zeta)$ is an analytic function of ζ in a neighborhood of every real point ζ, and the $R^\nu(\zeta)$ may be so chosen as to have the same property.

Proof: λ is a solution of the equation

$$\det(\lambda I - M) = G(\lambda,\zeta) = \lambda^k + g_{(k-1)}(\zeta)\lambda^{k-1} + \cdots + g_o(\zeta) = 0. \tag{1}$$

Here the g_ν are polynomials in ζ. Let us factor G into irreducible polynomials H of the form

$$H(\lambda,\zeta) = h_p(\zeta)\lambda^p + h_{(p-1)}(\zeta)\lambda^{p-1} + \cdots + h_o(\zeta). \tag{2}$$

Then each eigenvalue λ is a solution of an equation

$$H(\lambda,\zeta) = 0. \tag{3}$$

Thus λ is by definition an algebraic function of ζ. From the theory of algebraic functions*, we know that the equation $H(\lambda,\zeta_o) = 0$ has p distinct roots except at certain exceptional points ζ_o where either

1. h_p vanishes

or

2. the discriminant of the equation, which is a polynomial in ζ, vanishes.

The discriminant of an irreducible polynomial cannot vanish identically. Hence the singular points of type 2 are finite in number. Furthermore in

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[*] See, e.g., [8] for the statements about algebraic functions which we employ here.

our case there can be no real points of type 2, for at such points the multiplicity of some eigenvalue must be greater than at neighboring points. Also we observe that $h_p(\zeta)$ must be a constant, so there can be no points of type 1. Since algebraic functions are analytic except at exceptional points, the assertion for $\lambda(\zeta)$ follows.

The eigenvectors R are solutions of a linear algebraic system

$$TR = 0. \tag{4}$$

Here $T = (t_{ij}) = \lambda(\zeta)I - P - \zeta A$. Thus the t_{ij} are analytic functions of ζ for real ζ. The nullity of T is q, and if r denotes the rank of T, we have

$$r + q = k. \tag{5}$$

To solve (4) for the components of R we may first reduce it to an equivalent system of r equations and k unknowns. We then assign arbitrary values to k-r = q of the unknowns $x_1, \ldots, x_q$ such that the remaining $r \times r$ coefficient matrix is non-singular. We may then solve for the remaining r unknowns by Cramer's rule. The $x_{q+1}, \ldots, x_k$ will therefore be rational functions of the $x_1, \ldots, x_q$ and the t_{ij} with non-vanishing denominator. By choosing $x_1, \ldots, x_q$ as analytic functions of ζ we are guaranteed that R is an analytic function of ζ. We obtain q linearly independent solutions $R(\zeta)$, and the analyticity is preserved in the orthonormalization procedure to obtain the $R^1(\zeta), \ldots, R^q(\zeta)$.

Appendix 2

Definitions: Given a k-dimensional vector space V, a real symmetric operator A, and a subspace T. A is non-positive over T if $(u,Au) \leq 0$ for all u in T. A is positive over T if $(u,Au) > 0$ for all $u \neq 0$ in T. Let α_ν, r^ν denote (real) eigenvalues and orthonormal eigenvectors of A; i.e., $Ar^\nu = \alpha_\nu r^\nu$, $\nu = 1,\ldots,k$. Let $\alpha_\nu > 0$ for $\nu = 1,\ldots,m$; $\alpha_\nu \leq 0$ for $\nu = m+1,\ldots,k$.

Lemma 1: Let T, R be disjoint subspaces of V (i.e., if u is in T and in R then $u = 0$), then $\dim T + \dim R \leq k$.

Proof: The proof of this lemma is trivial.

Lemma 2: Let A be non-positive over T. Then $\dim T \leq k-m$.

Proof: Let R be the subspace spanned by $r_1,\ldots,r_m$. Then T and R are disjoint, for if u is in R and in T, we have

$$u = \sum_{\nu=1}^{m} c_\nu r^\nu;$$

$$Au = \sum c_\nu \alpha_\nu r^\nu;$$

$$(u,Au) = \sum c_\nu^2 \alpha_\nu \leq 0.$$

Hence $c_\nu = 0$; $\nu = 1,\ldots,m$. Therefore $u = 0$.

Lemma 3: Let T be a subspace of V of maximal dimension such that A is non-positive over T. Then $\dim T = k-m$.

Proof: Let T_1 be the subspace spanned by $r_{m+1},\ldots,r_k$. Then A is non-positive over T_1 and $\dim T_1 = k-m$.

Lemma 4: Let A be positive over P. Then $\dim P \leq m$.

Proof: Let T_1 be defined as above. Then T_1 and P are disjoint, for if $u \neq 0$ is in T_1 and in P, we have

$$u = \sum_{\nu=m+1}^{k} c_\nu r^\nu;$$

$$Au = \sum c_\nu \alpha_\nu r^\nu;$$

$$(u,Au) = \sum c_\nu^2 \alpha_\nu > 0,$$

which is a contradiction.

Part II

Asymptotic expansion of steady-state solutions of symmetric hyperbolic linear differential equations

Introduction

In a series of lectures[9] given at New York University during 1947-48, R. K. Luneburg introduced a method for obtaining asymptotic expansions of solutions of Maxwell's equations. The theory was refined and extended to linear second order hyperbolic partial differential equations by M. Kline[10]. Similar asymptotic expansions have been studied by several authors* and have been applied to the solution of diffraction problems[11]. These methods have the virtue of providing the asymptotic expansion of the solution even where it is difficult or impossible to obtain the solution itself. In cases where the exact solution is known, the expansion is frequently more useful and can be obtained more easily by these methods than by expanding the known solution.

The present paper is concerned with extending the Luneburg-Kline theory in two directions: First, we will consider more general partial differential equations. Second, we will attempt to provide proofs of certain assertions for which Luneburg and Kline use plausibility arguments. Our goal is a rigorous proof of the asymptotic nature of the formal expansions which will be derived. In this proof the results of Part I play a key role but nevertheless we have attempted to make the present discussion self-contained.

The plan of presentation is as follows: In the remainder of this introduction we discuss notation and the formulation of the problem. Then in Section 1 a formal expansion is derived. In Section 2 this expansion is applied to Maxwell's equations. Thus Sections 1 and 2 contain our formal results. The remaining sections are concerned with the proof of the asymptotic nature of these formal results.

We now turn to the definitions and formulation of the problem. A fuller discussion of some of the following notions is contained in Part I. We denote by u, a k component vector function of the (n+1) variables x_o, $x_1, \ldots, x_n$, and by A^ν and B, k x k matrices with real entries. We will sometimes denote x_o by t and $x_1, \ldots, x_n$ by x. The matrices A^ν and B are taken to be smooth* functions of x in the entire x-space. We define a first order linear differential operator

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* See the references listed in [11].

$$(1) \qquad Lu = \sum_{\nu=o}^{n} A^\nu u_{x_\nu} + Bu \qquad (u_{x_\nu} = \frac{\partial u}{\partial x_\nu}).$$

The operator L is taken to be symmetric hyperbolic, i.e., the matrices A^ν are assumed to be symmetric, and the matrix A^o to be positive-definite. Without loss of generality we may take $A^o = I$. Now we consider the initial value problem for the vector w(t,x)

$$(2) \qquad Lw = g(x)e^{-i\omega t} ; \qquad w(0,x) = h(x).$$

Here g and h are vector functions and ω is a (scalar) parameter which will be referred to as the (circular) frequency. In physical applications, the function $ge^{-i\omega t}$ represents a source with harmonic time dependence.

Under appropriate conditions (which will be made explicit later) the solution w of (2) may be decomposed uniquely into two terms:

$$(3) \qquad w(t,x) = z(t,x) + e^{-i\omega t} v(x)$$

where $z(t,x) \to 0$ as $t \to \infty$. z is called the transient, and v(x), will be referred to as the steady-state function. The problem we consider is that of obtaining the asymptotic expansion of v with respect to ω, as $\omega \to \infty$.

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* The term 'smooth' will be used to describe functions which have continuous derivatives of sufficiently high order for the purpose at hand.

It will be shown that $v(x)$ is independent of the initial function h and is a solution of the reduced equation,

$$(4) \qquad Nv = \sum_{\nu=1}^{n} A^{\nu} v_{x_{\nu}} + (B - i\omega)v = g.$$

(Eq. (4) may be obtained formally by substituting $w = e^{-i\omega t} v$ into eq. (2).) Although the reduced equation does not suffice to determine v uniquely, it can be used in formally deducing the expansion of v. In this deduction it is necessary to impose on the expansion an outgoing condition which will be explained later. In the later sections it will be shown that the formal expansion is indeed an asymptotic expansion (for $\omega \to \infty$) of $v(x)$.

1. The formal expansion

We seek a formal expansion $\tilde{v}$ of the steady state function v defined in the Introduction. This function satisfies the equation

$$(5) \qquad Nv = g(x)$$

Our procedure will be formal; the results will be justified later. We begin by considering an expansion of the form

$$(6) \qquad y = \sum_{r=o}^{\infty} \frac{y^{(r)}(x)}{(i\omega)^r} .$$

Here the $y^{(r)}(x)$ are vector functions to be determined. If we insert (6) in (5) and compare powers of ω, we obtain

$$(7) \qquad y^{(o)} = 0; \qquad y^{(1)} = -g; \qquad y^{(r+1)} = \mathcal{L} y^{(r)}; \qquad r = 1,2,\ldots .$$

The operator $\mathcal{L}$ is given by

$$(8) \qquad \mathcal{L} = \sum_{\nu=1}^{n} A^{\nu} \frac{\partial}{\partial x_{\nu}} + B.$$

From (7) we obtain the functions $y^{(r)}$ explicitly in the form

$$(9) \qquad y^{(o)} = 0; \qquad y^{(r)} = - \mathcal{L}^{(r-1)} g; \qquad r = 1,2,\ldots .$$

Here $\mathcal{L}^{o} = I$. Thus

$$(10) \qquad y = - \sum_{r=o}^{\infty} \frac{\mathcal{L}^{r} g}{(i\omega)^{r+1}} .$$

If g(x) is infinitely differentiable, the desired formal solution is simply

$$(11) \qquad \tilde{v} = y.$$

If the source region is bounded (i.e., if $g \equiv 0$ outside a bounded region) the formal solution vanishes identically outside that region.

Let us now consider the more interesting case in which g or one of its derivatives is discontinuous. More precisely, let Γ be a smooth (n-1) dimensional manifold in x-space, and let g(x) be smooth on both sides of Γ, while g and its derivatives have jump discontinuities across Γ. Later, in considering problems with boundaries, we shall require that g(x) vanish in a neighborhood of the boundary. A function g(x) with all these properties will be said to be of type g. We require that our formal solution v shall be continuous. (It will be seen later that the exact solution v is also continuous.) To achieve this we may begin with (10) but we must repair the discontinuities in it without disturbing the equation (5). We may hope to accomplish this by adding to y solutions of the homogeneous equation

$$Nv = 0. \tag{12}$$

For this purpose we shall study functions of the form

$$z^{(\alpha)} = e^{i\omega\Psi^{(\alpha)}(x)} \sum_{r=0}^{\infty} \frac{z^{(\alpha)(r)}(x)}{(i\omega)^r} \quad . \tag{13}$$

Here α is an index, $\Psi^{(\alpha)}$ is a scalar function, $z^{(\alpha)}$ and $z^{(\alpha)(r)}$ are vector functions*.

If we apply the operator N to (13) and equate the coefficient of each power of $(i\omega)$ to zero we obtain

$$G^{(\alpha)}\, z^{(\alpha)(0)} = 0 \tag{14}$$

$$G^{(\alpha)}\, z^{(\alpha)(r+1)} - \mathcal{L} z^{(\alpha)(r)} = 0; \qquad r = 0,1,2,\ldots \tag{15}$$

where

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* The formal expansion y defined by (6) is a special case of the form (13). In fact, if one attempts to satisfy the inhomogeneous equation (5) by inserting (13) one is forced to set $\Psi^{(\alpha)} \equiv 0$. Thus, our formal procedure requires the single "ansatz"(13).

(16) $$G^{(\alpha)} = I - \sum_{\nu=1}^{n} \Psi^{(\alpha)}_{x_\nu} A^\nu .$$

If the matrix $G^{(\alpha)}$ were non-singular, these equations would have only the trivial solution $z^{(\alpha)(r)} \equiv 0$; $r = 0,1,\ldots$. Hence we assume that

(17) $$\det G^{(\alpha)} = 0 .$$

Before taking a closer look at the equations (14), (15), it is necessary to impose an important condition on the class of operators L (or N) under consideration, which we shall call the

multiplicity condition: If $P(x,z) = \sum_{\nu=1}^{n} z_\nu A^\nu$, where the z_ν are real numbers and not all zero, then the multiplicities of the eigenvalues of the matrix P are independent of x and $z = z_1,\ldots,z_n$.

For convenience, we shall temporarily omit the index (α) in our consideration of a particular $z^{(\alpha)}$. From (17) we see that the matrix G is singular, so it has at least one zero eigenvalue at every point. At some point, let $\lambda_1,\ldots,\lambda_q$ denote the zero eigenvalues of G (i.e., $\lambda_1 = \lambda_2 = \cdots = \lambda_q = 0$). Then the multiplicity condition assures us that the eigenvalues $\lambda_1(x),\ldots,\lambda_q(x)$ of G(x) are zero everywhere, and all the others are non-zero*. Let $R^1,\ldots,R^q$ be an orthonormal set of null eigenvectors of G, i.e., those eigenvectors out of the complete set $R^1,\ldots,R^k$ of orthonormal eigenvectors which are associated with the eigenvalue $\lambda_1 \equiv \cdots \equiv \lambda_q \equiv 0$. From (14) we see that $z^{(0)}$ is in the null-space of G, so it has an expansion in terms of the null eigenvectors:

(18) $$z^{(0)} = \sum_{m=1}^{q} \sigma^0_m R^m$$

(the $\sigma^0_m = \sigma^0_m(x)$ are scalars). We obtain a differential equation for $z^{(0)}$ from (15) for r = 0 by multiplying (15) by a null eigenvector:

(19) $$\sum_{\nu=1}^{n} R^\ell A^\nu z^{(0)}_{x_\nu} + R^\ell B z^{(0)} = 0; \qquad \ell = 1,\ldots,q.$$

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* For a clarification of this assertion, see Part I, Section 2, the second footnote.

Inserting (18) in (19) we obtain

$$(20) \qquad \sum_{\nu=1}^{n}\sum_{m=1}^{q} R^{\ell} A^{\nu} R^{m} (\sigma^{o}_{m})_{x_\nu} + \sum_{\nu=1}^{n}\sum_{m=1}^{q} R^{\ell} A^{\nu} R^{m}_{x_\nu} (\sigma^{o}_{m}) + \sum_{m=1}^{q} R^{\ell} B R^{m} \sigma^{o}_{m} = 0; \quad \ell = 1,\dots,q.$$

Equation (20) is a first order system of linear differential equations for the scalar coefficients σ^{o}_{m} in the expansion of $z^{(o)}$. Before discussing it further, we will derive the corresponding equations for $z^{(r)}$, $r > 0$.

To do this we expand $z^{(r)}$ in terms of all the eigenvectors of G

$$(21) \qquad z^{(r)} = \sum_{m=1}^{k} \sigma^{r}_{m} R^{m}.$$

From (15) and (21) we have

$$(22) \qquad Gz^{(r)} = \sum_{\nu=1}^{n} A^{\nu} z^{(r-1)}_{x_\nu} + Bz^{(r-1)} = \sum_{m=q+1}^{k} \sigma^{r}_{m} \lambda_{m} R^{m}; \qquad r = 1,2,\dots \ .$$

Here the eigenvalue λ_m is simply

$$(23) \qquad \lambda_{m} = R^{m} G R^{m}.$$

Multiplying (22) be an eigenvector which is not null, we obtain

$$(24) \qquad \sigma^{r}_{s} = \frac{1}{\lambda_s} \left\{ \sum_{\nu=1}^{n} R^{s} A^{\nu} z^{(r-1)}_{x_\nu} + R^{s} B z^{(r-1)} \right\}; \qquad \begin{array}{l} s = q+1,\dots,k \\ r = 1,2,\dots \ . \end{array}$$

Thus we see that the coefficients of the non-null eigenvectors in the expansion (21) of $z^{(r)}$ are given explicitly by (24) in terms of $z^{(r-1)}$.

To obtain the coefficients of the null eigenvectors, we multiply (15) by a null eigenvector

$$(25) \qquad \sum_{\nu=1}^{n} R^{\ell} A^{\nu} z^{(r)}_{x_\nu} + R^{\ell} B z^{(r)} = 0; \qquad \begin{array}{l} \ell = 1,\dots,q \\ r = 0,1,2,\dots \ . \end{array}$$

Inserting (21) in (25) and rearranging terms we obtain

(26) $$\sum_{\nu=1}^{n}\sum_{m=1}^{q} R^{\ell} A^{\nu} R^{m} (\sigma^{r}_{m})_{x_\nu} + \sum_{m=1}^{q} \tau_{\ell m}\, \sigma^{r}_{m} = g^{r}_{\ell} \qquad \begin{array}{l} \ell = 1,\ldots,q \\ r = 0,1,2,\ldots \end{array}$$

where

(27) $$\tau_{\ell m} = \sum_{\nu=1}^{n} R^{\ell} A^{\nu} R^{m}_{x_\nu} + R^{\ell} B R^{m}$$

and

(28) $$g^{r}_{\ell} = -\sum_{\nu=1}^{n}\sum_{s=q+1}^{k} R^{\ell} A^{\nu} R^{s} (\sigma^{r}_{s})_{x_\nu} - \sum_{s=q+1}^{k} \tau_{\ell s}\, \sigma^{r}_{s} .$$

By virtue of (24), g^{r}_{ℓ} is given explicitly in terms of $z^{(r-1)}$. If we establish the convention $z^{(-1)} = 0$ (20) appears as a special case of (26). (This has been anticipated by including $r = 0$ in (26).)

It is possible to put (26) in a much simpler form. For this purpose we employ the following lemma which is a consequence of the multiplicity condition[*].

> Lemma: Let $R^{1}_{o},\ldots,R^{q}_{o}$ be eigenvectors associated with the same eigenvalue λ_{o} of the matrix $P = \sum_{\nu=1}^{n} z_{\nu} A^{\nu}$ (z_ν real). Then for each j,

(29) $$R^{\ell}_{o}\ A^{j} R^{m}_{o} = c_{j}\, \delta_{\ell m}$$

(i.e., $R^{\ell}_{o} A^{j} R^{m}_{o} = 0$ if $\ell \neq m$, and is independent of ℓ if $\ell = m$.)

It follows immediately from the lemma that (26) may be written

(30) $$\sum_{\nu=1}^{n} R^{1} A^{\nu} R^{1} (\sigma^{r}_{\ell})_{x\nu} + \sum_{m=1}^{q} \tau_{\ell m}\, \sigma^{r}_{m} = g^{r}_{\ell} \qquad \begin{array}{l} \ell = 1,\ldots,q \\ r = 0,1,2,\ldots \end{array} .$$

We now introduce a family of curves in the x-space defined parametrically by $x_{\nu} = x_{\nu}(s)$, where

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[*] The proof appears in Part I, Section 2.

$$(31) \qquad \dot{x}_\nu = \frac{dx_\nu}{ds} = R^1 A^\nu R^1; \qquad \nu = 1,\ldots,n.$$

These curves will be referred to simply as rays. The equation (30) may be written as an ordinary differential equation along a ray:

$$(32) \qquad \dot{\sigma}_\ell^r + \sum_{m=1}^{q} \tau_{\ell m}\, \sigma_m^r = g_\ell^r \qquad \begin{array}{l} \ell = 1,\ldots,q \\ r = 0,1,2,\ldots\ . \end{array}$$

For each fixed r, (32) is a system of q linear ordinary differential equations. The recursive system of differential equations (32), together with the algebraic equations (24) will be called the transport equations.

The function Ψ appearing in the expansion (13) of $z^{(\alpha)}$ is called the phase function. We have seen that it is a solution of the single first order non-linear partial differential equation (17), a special case of which is called the eiconal equation. It is interesting to note, and not difficult to verify that our 'rays' are the 'characteristic curves' used in constructing the solution of the first order equation for Ψ. We also note that

$$(33) \qquad \frac{d\Psi}{ds} = \sum_{\nu=1}^{n} \Psi_{x_\nu} \dot{x}_\nu = \sum_{\nu=1}^{n} R^1 \Psi_{x_\nu} A^\nu R^1 = R^1 \left[I-G \right] R^1 = 1.$$

Thus Ψ may be identified (up to an additive constant on each ray) with the curve parameter s.

A formal series $z^{(\alpha)}$ of the form (13), where $\Psi^{(\alpha)}$ is a solution of the eiconal equation, and the $z^{(\alpha)(r)}$ (or rather their scalar components) are solutions of the transport equations will be called a wave. A wave is not completely determined until we specify initial conditions for the eiconal equation and for the transport equations. Thus we shall, in general, have some (n-1)-dimensional manifold, on which the value of Ψ is prescribed. Because of the non-linearity of the eiconal equation (17) there may be several solutions with these prescribed initial values. We shall think of each solution Ψ as defined only on one side of the initial manifold (and zero on the other), and in its region of definition we associate with Ψ a corresponding wave (for which the initial values of the $z^{(r)}$ still need to be prescribed) which is also zero outside this region. Thus our waves will in general be discontinuous across their initial manifolds.

It is, by now, perhaps obvious what must be done to complete the construction of $\tilde{v}$. We shall attempt to repair the discontinuities in y by adding waves to (10). Thus our formal solution will be

$$\tilde{v} = y + \sum_{\alpha} z^{(\alpha)} \tag{34}$$

where the $z^{(\alpha)}$ must be appropriately specified. Since y is smooth except on the manifold Γ, we take Γ to be the initial manifold for each $z^{(\alpha)}$. At this point we can easily state the *outgoing condition* referred to in the Introduction. First we note that with every solution Ψ, and therefore with every wave, is associated an (n-1) parameter family of rays. A wave will be said to be *outgoing* (*incoming*) with respect to the initial manifold Γ if on every one of its associated rays, Ψ increases (decreases) monotonically with distance from Γ. In the expansion of (34) only waves $z^{(\alpha)}$ which are outgoing with respect to Γ are to be included.

The requirement that v shall be continuous may be stated simply as

$$[\tilde{v}]^{\Gamma} = 0 \qquad \text{on } \Gamma, \tag{35}$$

where $[\tilde{v}]^{\Gamma}$ denotes the jump in $\tilde{v}$ across Γ (i.e., the difference of the limiting values as Γ is approached from each side) with some given orientation. Inserting (34) in (35), we have

$$[y]^{\Gamma} + \sum_{\alpha} e(\alpha)\, z^{(\alpha)} = 0 \qquad \text{on } \Gamma. \tag{36}$$

Here $e(\alpha) = \pm 1$ and it is determined by the orientation of $[\;]^{\Gamma}$ and the side of Γ on which $z^{(\alpha)}$ is zero. Since $[y]^{\Gamma}$ contains no exponential factor in ω we see that (36) can be satisfied only if

(37) $$\Psi^{(\alpha)} = 0 \quad \text{on } \Gamma.$$

Inserting (6) and (13) in (36) and equating the coefficient of each power of $(i\omega)$ to zero, we obtain

(38) $$\sum_{\alpha} e(\alpha) z^{(\alpha)(0)} = 0 \quad \text{on } \Gamma$$

(39) $$\sum_{\alpha} e(\alpha) z^{(\alpha)(r)} = \left[\mathcal{L}^{(r-1)} g\right]^{\Gamma}, \quad \text{on } \Gamma; \quad r = 1, 2, \ldots \quad .$$

Now we must determine the range of α, that is we must determine how many outgoing waves appear in (34). In solving the eiconal equation and initial conditions (37), $\Psi^{(\alpha)}$ and its derivatives must satisfy the so-called 'strip manifold condition'. Geometrically this means that the solution surface $\Psi = \Psi^{(\alpha)}(x)$ in Ψx-space must be tangent to Γ, so the projection into x-space of the normal to the solution surface must be parallel to the normal to Γ. Thus

(40) $$\Psi^{(\alpha)}_{x_\nu} = \rho(\alpha)\gamma_\nu ; \qquad \nu = 1, \ldots, n$$

where $\gamma_1, \ldots, \gamma_n$ are components of the unit normal to Γ in x-space, and $\rho(\alpha)$ is a proportionality factor. It is easily seen from the eiconal equation that a solution surface can never have a tangent plane parallel to the plane $\Psi = 0$ (in Ψx-space). Hence each solution $\Psi^{(\alpha)}$ is increasing (with distance) on one side of Γ and decreasing on the other. Thus each $\Psi^{(\alpha)}$ provides us with two waves, one on each side of Γ. One is outgoing and one is incoming. We use only the outgoing one. The scalar factor $\rho(\alpha)$ is determined by the eiconal equation (17); thus

(41) $$\det G^{(\alpha)} = \det\Big(I - \sum_{\nu=1}^{n} \Psi^{(\alpha)}_{x_\nu} A^{\nu}\Big) = \det(I - \rho(\alpha)E) = 0, \quad \text{on } \Gamma.$$

where

$$E = \sum_{\nu=1}^{n} \gamma_\nu A^\nu. \tag{42}$$

There will be as many outgoing waves as there are distinct solutions $\rho(\alpha) = \frac{1}{\beta(\alpha)}$ of (41), hence as many as there are distinct non-zero eigenvalues $\beta(\alpha)$ of E. (There will be equally many incoming waves, namely the waves associated with the continuation of each $Y^{(\alpha)}$ across Γ.)

Let us introduce the orthonormal eigenvectors B^j of E which are associated with the non-zero eigenvalues $\beta(j)$:

$$(I - \rho(j)E)B^j = G^{(j)}B^j = 0, \quad \text{on } \Gamma. \tag{43}$$

The $\rho(j)$ and $G^{(j)}$ in (43) are not necessarily distinct for distinct j, i.e., $\rho(j) = \rho(i)$ and $G^{(j)} = G^{(i)}$ if $\beta(j) = \beta(i)$. Furthermore each non-null eigenfector B^j of F is a null eigenvector of $G^{(\alpha)}$ for some α, and the totality of null eigenvectors for all the $G^{(\alpha)}$ are the totality of non-null eigenvectors B^j of E. This statement holds, of course only on Γ where the B^j are defined. The matrix E, however, may well be singular. In this case there will also be null eigenvectors $J^1,\ldots,J^p$ of E. In order to complete the construction of a continuous formal solution v it is necessary to impose a new condition on the discontinuities of g(x) which we call the

source condition: $\quad J^\nu[g]^\Gamma = 0; \quad \nu = 1,\ldots,p; \quad \text{on } \Gamma.$

(Here we have, of course, an inner-product of two vectors. If E is non-singular the condition is empty, i.e., it does not restrict g.) It will be seen in the next section that the source condition has a very interesting interpretation for Maxwell's equations.

We must now proceed to select initial values in order to satisfy (38) and (39). In so doing we will make constant use of the fact that the set of eigenvectors B^j may be decomposed into disjoint subsets each of which is precisely the set of null eigenvectors of one of the $G^{(\alpha)}$. Since the B^j are independent and since each $z^{(\alpha)(0)}$ has an expansion in terms of the null eigenvectors of $G^{(\alpha)}$, it follows from (38) that each $z^{(\alpha)(0)}$ is zero on Γ. Therefore the initial values of the $\sigma_m^{o(\alpha)}$ are zero, and since equation (32) is homogeneous for $r = 0$, we conclude that $z^{(\alpha)(0)} \equiv 0$ for each α*. We now consider (39) for $r = 1$. Again each $z^{(\alpha)(1)}$ has an expansion in terms of the null eigenvectors of $G^{(\alpha)}$, for by (24) $\sigma_s^{1(\alpha)} = 0$ because $z^{(\alpha)(0)} \equiv 0$. It is clear now that we may satisfy (39) only because of the source condition, and that the initial values on Γ of each $z^{(\alpha)(1)}$ are uniquely determined. Now suppose we have determined the $z^{(\alpha)(\rho)}$ and have satisfied (39) for $\rho = 0,1,\ldots,r-1$. We recall that $z^{(\alpha)(r)}$ has an expansion in terms of all the eigenvectors of $G^{(\alpha)}$. By (24) the coefficients of the non-null eigenvectors are determined in terms of $z^{(\alpha)(r-1)}$. It remains to determine the coefficients of the null eigenvectors from (39). This can be done, and (39) will be satisfied if and only if

(44) $$\sum_{\alpha} e(\alpha) J^{\nu} z^{(\alpha)(r)} = J^{\nu}\left[\mathcal{L}^{(r-1)} g\right]^{\Gamma}, \quad \text{on } \Gamma; \quad \nu = 1,\ldots,p.$$

We will now show that (44) is a consequence of the foregoing induction hypothesis. First we consider the directional derivative

(45) $$\frac{d}{d\tau} = \sum_{j=1}^{n} J^{\nu} A^{j} \frac{\partial}{\partial x_j},$$

for any $\nu = 1,\ldots,p$. It is easily seen that the direction of $\frac{d}{d\tau}$ is tangential to Γ. Hence for any vector u which is smooth on both sides of Γ,

(46) $$\frac{d}{d\tau}\,[u]^{\Gamma} = \left[\frac{d}{d\tau} u\right]^{\Gamma}.$$

Hence

(47) $$J^{\nu}\mathcal{L}[u]^{\Gamma} = \frac{d}{d\tau}[u]^{\Gamma} + J^{\nu}B[u]^{\Gamma} = \left[\frac{d}{d\tau} u\right]^{\Gamma} + \left[J^{\nu}Bu\right]^{\Gamma} = \left[J^{\nu}\mathcal{L} u\right]^{\Gamma} = J^{\nu}\left[\mathcal{L} u\right]^{\Gamma}.$$

It follows that

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* i.e., we could have started the expansion (13) with $r = 1$

$$J^{\nu}\mathcal{L}\left[\mathcal{L}^{(r-2)}g\right]^{\Gamma} = J^{\nu}\left[\mathcal{L}^{(r-1)}g\right]^{\Gamma}. \tag{48}$$

We observe that each J^{ν} is a non-null eigenvector for every $G^{(\alpha)}$. Thus from (24),

$$J^{\nu}z^{(\alpha)(r)} = \frac{1}{J^{\nu}G^{(\alpha)}J^{\nu}}\left\{ J^{\nu}\mathcal{L}z^{(\alpha)(r-1)} \right\}. \tag{49}$$

But

$$J^{\nu}G^{(\alpha)}J^{\nu} = J^{\nu}(I-\rho(\alpha)E)J^{\nu} = 1. \tag{50}$$

Hence

$$J^{\nu}z^{(\alpha)(r)} = J^{\nu}\mathcal{L}z^{(\alpha)(r-1)}. \tag{51}$$

By the induction hypothesis we may assume the validity of (39) with r replaced by r-1, i.e.,

$$\sum_{\alpha} e(\alpha)z^{(\alpha)(r-1)} = \left[\mathcal{L}^{(r-2)}g\right]^{\Gamma}. \tag{52}$$

Now using (51), (52), and (48), (in that order) we have

$$\begin{aligned}\sum_{\alpha} e(\alpha)J^{\nu}z^{(\alpha)(r)} &= J^{\nu}\mathcal{L}\sum_{\alpha} e(\alpha)z^{(\alpha)(r-1)} = J^{\nu}\mathcal{L}\left[\mathcal{L}^{(r-2)}g\right]^{\Gamma} \\ &= J^{\nu}\left[\mathcal{L}^{(r-1)}g\right]^{\Gamma}.\end{aligned} \tag{53}$$

Thus we have verified (44) and the induction argument is complete.

It follows that the initial values for $z^{(\alpha)(r)}$ will be uniquely determined for all α and r and (39) will be satisfied for all r. We may then solve the transport equations recursively for $r = 0,1,\ldots$ along each ray, starting at Γ, and determine the $z^{(\alpha)(r)}$. With this, the formal construction is complete except for the following reservation: Examination of the transport equations reveals that the coefficients of the linear differential equation (32) depend on the eigenvectors of the matrix $G^{(\alpha)}$, hence on the function $\Psi^{(\alpha)}$. By our assumptions about Γ, $\Psi^{(\alpha)}$ will be smooth in a neighborhood of Γ, but as we proceed along a ray from Γ we may reach a point where $\Psi^{(\alpha)}$ or one of its derivatives becomes singular. We cannot, by the present construction, continue the solutions of the transport equations beyond such a point. Points of this type are said to lie on a caustic*. For convenience we state a condition which guarantees that the construction can be completed:

caustic condition: The problem is such that the phase functions $\Psi^{(\alpha)}$, associated with each of the waves in the construction, are free of caustics.

Having completed the formal construction for the problem as formulated in the Introduction, we shall now show how to extend the construction to problems involving boundaries.

If $\beta(x) = 0$ is the equation of any smooth surface B in the x-space, the transformation $z_\nu = \alpha_\nu(x)$; $\nu = 1,\ldots,n$; with $\alpha_n = \beta$ maps B into the hyperplane $z_n = 0$, while preserving the symmetric hyperbolic nature of the operator L, and therefore the properties of the reduced operator N. Thus, without loss of generality we may take our boundary B to be the plane $x_n = 0$. On this boundary we require our solution v (hence our formal solution $\tilde{v}$) to satisfy the following type of homogeneous boundary condition.

The vector v is an element of a k-dimensional Euclidean vector space V. Consider a subspace T with the following properties:

1. The matrix A^n is non-positive over T, i.e., $uA^n u \leq 0$ for every u in T.

2. T is maximal, i.e., the dimension of T is as large as the dimension of any subspace having property 1.

Our boundary condition may now be expressed simply as

$$v(x) \text{ in } T, \quad \text{for } x_n = 0. \tag{54}$$

---

[*] For a discussion of the formal construction for certain problems involving caustics, see [12].

If $T^{\perp}$ is the orthogonal complement of T, and $T^{\perp}$ is generated by the vectors $b^1,\ldots,b^m$; the condition may also be stated as

$$vb^s = 0; \qquad s = 1,\ldots,m; \qquad \text{for } x_n = 0. \tag{55}$$

The vectors b^{ν} are assumed to be smooth functions of $x_1,\ldots,x_{n-1}$.

For problems with boundaries, we have assumed that the source region does not intersect the boundary, i.e., that $g(x) = 0$ in a neighborhood of the boundary. Thus the term y in the expansion (20) of $\tilde{v}$ automatically satisfies the boundary condition. Suppose now that a particular wave $z^{(\alpha)}$ (outgoing from Γ) is 'incident' on the boundary, B (i.e., any of the associated rays intersect the boundary). We observe first that $z^{(\alpha)}$ will be incoming with respect to B. Our procedure will be to add appropriate reflected waves to the expansion $\tilde{v}$ which will be outgoing with respect to B, and which, together with $z^{(\alpha)}$ will satisfy the boundary condition. We emphasize that the condition will be satisfied individually for each $z^{(\alpha)}$ together with its associated reflected waves. We shall see that the boundary condition uniquely determines the reflected waves.

The incoming wave $z^{(\alpha)}$ will have some given phase (i.e., the value of $\Psi^{(\alpha)}$) on B. It is clear that the associated reflected waves must have the same phase on B. In geometrical language, if the phase surface $\Psi = \Psi^{(\alpha)}(x)$ in Ψx-space intersects the boundary B (i.e., the manifold $x_n = 0$ in Ψx-space) in the $(n-1)$-dimensional manifold $\triangle$, then the corresponding phase surfaces of the reflected waves must also intersect B in $\triangle$. We shall sometimes denote by x_0 the coordinate Ψ in Ψx-space.

Let $\delta_0,\ldots,\delta_{n-1},0$ denote components of the unit normal to $\triangle$ in the plane $x_n = 0$. Let $\Psi = \Psi^{(p)}(x)$ be the equation of a phase surface $C^{(p)}$ whose intersection with B is $\triangle$. Then (by the strip manifold condition), since the components of the normal to $C^{(p)}$ are $(1, -\Psi^{(p)}_{x_1}, \ldots, -\Psi^{(p)}_{x_n})$, we have $1 = \rho\delta_0$; $-\Psi^{(p)}_{x_\nu} = \rho\delta_\nu$, $\nu = 1,\ldots,n-1$; and $\Psi^{(p)}_{x_n}$ is a solution of

$$\det G^{(p)} = \det\Big(I - \sum_{\nu=1}^{n} \Psi^{(p)}_{x_\nu} A^\nu\Big) = \det(-\Psi^{(p)}_{x_n} A^n + \rho F) = 0, \text{ on } \triangle \quad . \tag{56}$$

Here

$$F = \sum_{\nu=0}^{n-1} \delta_\nu A^\nu. \tag{57}$$

Equation (56) will, in general, have complex roots $\Psi^{(p)}_{x_n}$. For each distinct real root there will be a corresponding phase surface passing through $\triangle$.

We now impose a new condition on the problem, the

shadow condition: F shall be non-singular throughout B.

In Part I, Section 6[*] it is pointed out that at the point of minimum $\Psi^{(\alpha)}$, $F = I$. Therefore it follows from the above condition that F is positive definite on B. It is also pointed out that the condition guarantees that no rays of either the incoming or reflected waves can be tangent to the boundary. Thus we can have no shadows in the problem.

Since F is positive definite we may set $F = JJ'$ where J is a non-singular matrix hence has an inverse, S. Let $\mathcal{G} = SA^nS'$. Then $\mathcal{G}$ is symmetric and we may write

$$(\mathcal{G} - \alpha_p)\, Q^{(p)} = 0; \qquad p = 1,\ldots,k \tag{58}$$

Here the $Q^{(p)}$ are a complete set of orthogonal eigenvectors of $\mathcal{G}$, which we may normalize by the condition

$$Q^{(p)}SS'\, Q^{(p)} = 1; \quad p = 1,\ldots,k; \tag{59}$$

The eigenvalues α_p are real. Set $B^{(p)} = S'Q^{(p)}$; $p = 1,\ldots,k$. It follows from

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[*] The discussion of that section closely parallels this one if we identify Ψ with t, 'phase surface' with 'characteristic surface', etc.

(59) that the $B^{(p)}$ are unit vectors. They are clearly linearly independent. From (58), $J(SA^nS' - \alpha_p)J'B^{(p)} = 0$, or

$$(A^n - \alpha_p F)B^{(p)} = 0; \qquad p = 1,\ldots,k. \tag{60}$$

For each non-zero α_p, set $\Psi_{x_n}^{(p)} = \rho/\alpha_p$; then

$$G^{(p)}B^{(p)} = (-\Psi_{x_n}^{(p)}A^n + \rho F)B^{(p)} = 0; \quad \text{if } \alpha_p \neq 0. \tag{61}$$

Equation (61) expresses the fact that for each distinct non-zero α_p there is a distinct phase surface, $C^{(p)}$, passing through $\triangle$, and $B^{(p)}$ is a null eigenvector of the associated matrix $G^{(p)}$. Since we may have multiple roots α_p, we may have several distinct $B^{(p)}$ associated with the same matrix $G^{(p)}$.

We have up until now not distinguished between waves $z^{(p)}$ which are incoming and outgoing with respect to B*. For convenience we simply assign the value zero to the coefficients of all incoming waves except the one $z^{(\alpha)}$ which "arises" at Γ. We will then attempt to determine the initial values for the outgoing waves from the boundary condition:

$$b^\nu \sum_p z^{(p)} = 0, \quad \text{on } B; \qquad \nu = 1,\ldots,m. \tag{62}$$

Comparing powers of $(i\omega)$ in the expansions of $z^{(p)}$ it follows at once that

$$b^\nu \sum_p z^{(p)(r)} = 0; \text{ on } B; \qquad \begin{array}{l} \nu = 1,\ldots,m \\ r = 0,1,2,\ldots \end{array} \tag{63}$$

We recall that each $z^{(p)(0)}$ has an expansion in terms of null eigenvectors of $G^{(p)}$. But this set of vectors is just a subset of the set of vectors $B^{(j)}$. Thus

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* We are, of course, dealing only with waves on one side of B, i.e., within the region of the problem, which we take to be $x_n \geq 0$.

$$\sum_{p} z^{(p)(0)} = \sum_{p=1}^{k} \sigma_p^o B^{(p)}. \tag{64}$$

(The σ_p^o may be identified with the σ_m^o in (18), but we have renumbered them and (64) contains the coefficients for several different waves.) Inserting (64) in (63) for $r = 0$,

$$\sum_{p=1}^{k} b^{\nu} B^{(p)} \sigma_p^o = 0; \qquad \nu = 1,\ldots,m . \tag{65}$$

Of course, some of the σ_p^o are already determined, namely those associated with the incoming waves. It is easy to see that $\sigma_p^o \left\{\begin{matrix}\text{will}\\ \text{will not}\end{matrix}\right\}$ have a preassigned value at a point on B if Ψ $\left\{\begin{matrix}\text{decreases}\\ \text{increases}\end{matrix}\right\}$ with x_n, i.e., if

$$\dot{x}_n = \frac{dx_n}{ds} = \frac{dx_n}{d\Psi} = B^{(p)}{}_{A} n_{B}{}^{(p)} \quad \text{is} \quad \left\{\begin{matrix} < 0 \\ > 0 \end{matrix}\right\}.$$

(The possibility $\dot{x}_n = 0$ is precluded by the shadow condition.) Let $B^{(p)}{}_{A} n_{B}{}^{(p)}$ be $\left\{\begin{matrix} > 0 \text{ for } p = 1,\ldots,s \\ \leq 0 \text{ for } p = s+1,\ldots,k \end{matrix}\right\}$. We rewrite (65) placing known quantities (which we denote simply by W_j) on the right side:

$$\sum_{p=1}^{s} b^{\nu} B^{(p)} \sigma_p^o = W_1; \qquad \nu = 1,\ldots,m. \tag{66}$$

In Part I, Section 6 it is proved that the coefficient matrix $(b^{\nu} B^{(p)})$ of (66) is square (i.e., $s = m$) and non-singular. Hence (66) can be solved uniquely for $\sigma_1^o,\ldots,\sigma_m^o$; and these quantities provide the initial values for the transport equations of order zero ($r = 0$) for the reflected waves.

To obtain the initial conditions for the higher transport equations, we use (63) for $r = 1,2,\ldots$. We recall that each $z^{(p)(r)}$ has an expansion in terms of the eigenvectors of $G^{(p)}$, the coefficients of the non-null eigenvectors being known from (24) in terms of $z^{(p)(r-1)}$. Thus we may write

$$\sum_{p} z^{(p)(r)} = \sum_{p=1}^{k} \sigma_p^r B^{(p)} + W_2 . \tag{67}$$

From (63)

$$\sum_{p=1}^{k} b^{\nu}B^{(p)}\,\sigma_p^r = W_3, \qquad \nu = 1,\ldots,m. \tag{68}$$

As before, those σ_p^r associated with incoming waves are known. Thus

$$\sum_{p=1}^{s} b^{\nu}B^{(p)}\,\sigma_p^r = W_4, \qquad \nu = 1,\ldots,m\,. \tag{69}$$

Since (66) and (69) have the same coefficient matrix it is clear that the latter also has a unique solution. Thus the initial values for the transport equations of all orders (for the reflected waves) are uniquely determined.

Assuming, now, that the caustic condition is satisfied, we may solve the transport equations and completely determine all of the reflected waves (for all of the incident waves $z^{(\alpha)}$). This completes the construction of the formal solution for problems with boundaries*. If we now use the index α to label all our waves, including reflected ones, the formal expansion may be written in the form

$$\tilde{v} = -\sum_{r=0}^{\infty} \frac{\mathcal{L}^r g}{(i\omega)^{r+1}} + \sum_{\alpha} e^{i\omega\Psi^{(\alpha)}(x)} \sum_{r=1}^{\infty} \frac{z^{(\alpha)(r)}(x)}{(i\omega)^r}\,. \tag{70}$$

2. Maxwell's equations

In the present section we will demonstrate the applicability of the results of this paper to Maxwell's equations. We begin by stating the equations in their usual form:

$$\nabla \times H - \frac{1}{c}\frac{\partial}{\partial t}(\varepsilon E) = \frac{4\pi}{c} I; \qquad I = \sigma E + \frac{c}{4\pi} F(x,t) \tag{71}$$

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* For problems with multiple boundaries and multiple reflections the construction is an obvious extension of the foregoing.

(72) $$\nabla \times E + \frac{1}{c}\frac{\partial}{\partial t}(\mu H) = 0$$

(73) $$\nabla \cdot (\varepsilon E) = 4\pi\rho$$

(74) $$\nabla \cdot (\mu H) = 0.$$

Here the independent variables are $t = x_0$ and $x = (x_1,x_2,x_3)$; H, E, I, F are vectors with three components. The scalars ε, μ, σ will be taken to be smooth functions of x; ε and μ are assumed to be everywhere positive. (Physically we are restricting consideration to isotropic media in which the electromagnetic properties do not change with time, and vary smoothly with position.) F represents impressed currents, i.e., sources.

We shall show that equations (71) and (72) form a symmetric hyperbolic system. As such, the usual existence and uniqueness theorems for initial value problems (see Part I) apply. In view of the completeness of this theory equations (73), (74) appear to be somehow superfluous. The confusion is easily resolved, however, if we interpret them as additional conditions on the initial data of the problem. From this point of view, we begin with (71) and (72); then we define the function $\rho(x,t)$ by equation (73), and impose the following condition on the initial data:

(75) $$\nabla \cdot (\mu H) = 0 \quad \text{at} \quad t = 0.$$

It follows easily from (72) that

(76) $$\frac{\partial}{\partial t}(\nabla \cdot \mu H) = 0,$$

and (74) is an immediate consequence of (75) and (76). From (71) and (73) we obtain the 'equation of continuity',

(77) $$\nabla \cdot I + \rho_t = 0.$$

We now set $E = (\phi_1, \phi_2, \phi_3)$, $H = (\phi_4, \phi_5, \phi_6)$, $F = (f_1, f_2, f_3)$,

$$\phi = \begin{bmatrix} \phi_1 \\ \cdot \\ \cdot \\ \cdot \\ \vdots \\ \phi_6 \end{bmatrix}, \qquad f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then (71) and (72) may be written as

$$C^0 \phi_t + \sum_{\nu=1}^{3} C^\nu \phi_{x_\nu} + \mathcal{M} \phi + f = 0 \tag{78}$$

where the matrices C^0 and $\mathcal{M}$ are given by:

$$C^0 = \mathcal{D}^2, \quad \mathcal{D} = \begin{pmatrix} \sqrt{\frac{\varepsilon}{c}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{\varepsilon}{c}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\varepsilon}{c}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{\mu}{c}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{\mu}{c}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{\mu}{c}} \end{pmatrix}; \quad \mathcal{M} = \frac{4\pi\sigma}{c} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{79}$$

and the matrices C^ν may be conveniently written together in the form:

$$\mathcal{H} = \sum_{\nu=1}^{3} z_\nu C^\nu = \begin{pmatrix} 0 & 0 & 0 & 0 & z_3 & -z_2 \\ 0 & 0 & 0 & -z_3 & 0 & z_1 \\ 0 & 0 & 0 & z_2 & -z_1 & 0 \\ 0 & -z_3 & z_2 & 0 & 0 & 0 \\ z_3 & 0 & -z_1 & 0 & 0 & 0 \\ -z_2 & z_1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

To transform (78) into our standard form, we set $\emptyset = \mathcal{D}^{-1}u$. If we insert this expression in (78) and multiply on the left by $\mathcal{D}^{-1}$ we obtain

$$(80) \qquad u_t + \sum_{\nu=1}^{3} A^{\nu} u_{x_\nu} + Bu = \eta$$

where

$$(81) \qquad A^{\nu} = \mathcal{D}^{-1} C^{\nu} \mathcal{D}^{-1}$$

$$(82) \qquad B = \sum_{\nu=1}^{3} \mathcal{D}^{-1} C^{\nu} (\mathcal{D}^{-1})_{x_\nu} + \mathcal{D}^{-1} \mathcal{M} \mathcal{D}^{-1} = \sum_{\nu=1}^{3} A^{\nu} \mathcal{D} (\mathcal{D}^{-1})_{x_\nu} + \mathcal{D}^{-1} \mathcal{M} \mathcal{D}^{-1}$$

$$(83) \qquad \eta = - \mathcal{D}^{-1} f.$$

It is clear from either (78) or (80) that we have a symmetric hyperbolic system. Furthermore if we define the matrix

$$(84) \qquad P = \sum_{\nu=1}^{3} z_\nu A^{\nu} = \mathcal{D}^{-1} \mathcal{H} \mathcal{D}^{-1} = e \mathcal{H} ; \qquad e = \frac{c}{\sqrt{\varepsilon\mu}}$$

one may verify that the eigenvalues of P are

$$(85) \qquad 0, 0, ed, ed, -ed, -ed ; \qquad d = \sqrt{z_1^2 + z_2^2 + z_3^2} .$$

It follows at once that the multiplicity condition is satisfied.

For convenience, we also list a set of orthonormal eigenvectors of P in the same order as their corresponding eigenvalues appear in (85):

$$(86) \qquad R^1 = \frac{1}{d\sqrt{2}} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}, \qquad R^2 = \frac{1}{d\sqrt{2}} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ -z_1 \\ -z_2 \\ -z_3 \end{bmatrix},$$

(86 cont'd.)

$$R^3 = \frac{1}{d\sqrt{2(z_1^2+z_2^2)}} \begin{bmatrix} -z_1z_3 \\ -z_2z_3 \\ z_1^2+z_2^2 \\ dz_2 \\ -dz_1 \\ 0 \end{bmatrix}, \qquad R^4 = \frac{1}{d\sqrt{2(z_1^2+z_3^2)}} \begin{bmatrix} dz_2 \\ -dz_1 \\ 0 \\ z_1z_3 \\ z_2z_3 \\ -(z_1^2+z_2^2) \end{bmatrix},$$

$$R^5 = \frac{1}{d\sqrt{2(z_1^2+z_2^2)}} \begin{bmatrix} z_1z_3 \\ z_2z_3 \\ -(z_1^2+z_2^2) \\ z_2d \\ -z_1d \\ 0 \end{bmatrix}, \qquad R^6 = \frac{1}{d\sqrt{2(z_1^2+z_2^2)}} \begin{bmatrix} dz_2 \\ -dz_1 \\ 0 \\ -z_1z_3 \\ -z_2z_3 \\ z_1^2+z_2^2 \end{bmatrix}.$$

For Maxwell's equations, we shall take our boundaries to be 'perfect conductors', i.e., we shall apply the boundary condition

$$(87) \qquad E_{tan} = 0$$

where F_{tan} denotes the components of E tangential to the boundary. If we denote the boundary by $y_3(x) = 0$, the transformation $y_\nu = y_\nu(x)$; $\nu = 1,2,3$ leads to an equation in which the coefficient matrix $\hat{A}_3$ multiplying u_{y_3} is simply the matrix $P = e\mathcal{H}$, provided we set $z_\nu = \frac{\partial y_3}{\partial x_\nu}$. Since $n = (z_1,z_2,z_3)$ now denotes a vector normal to the boundary, the condition may be expressed as

$$(88) \qquad u_j = cz_j; \qquad j = 1,2,3$$

where u_j denotes components of u. It is clear that all vectors u satisfying

(88) lie in a space T of dimension 4. Furthermore, for u in T it is easily verified that $\mu \hat{A}^3 u = 0$. Thus $\hat{A}^3$ is non-positive over T. Since $\hat{A}^3 = P$ has precisely 4 non-positive eigenvalues, it is easily shown (see Part I, Appendix 2) that T is maximal. Thus we have shown that the boundary condition (87) is of the form we have applied in Section 1.

We examine, now, the 'source condition', which was introduced in Section 1. If $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ denotes a unit vector normal to the surface of discontinuity Γ in the source region, and J^μ are null eigenvectors of the matrix $E = \sum_{\nu=1}^{3} \gamma_\nu A^\nu$, this condition is

$$(89) \qquad J^\mu [g] = 0 \quad \text{(for all } J^\mu\text{)}.$$

The function g(x) is determined by

$$(90) \qquad g e^{-i\omega t} = \eta = - \mathcal{S}^{-1} f = - \sqrt{\frac{c}{\varepsilon}}\, f.$$

We may identify the matrix E with the matrix P by setting $\gamma_\nu = z_\nu$; $\nu = 1,2,3$. Thus the J^μ are given by R^1 and R^2 of (86). Since ε is a continuous function, we may rewrite (89) as

$$R^\mu [f] = 0; \qquad \mu = 1,2;$$

or in terms of 3 component vectors,

$$(91) \qquad \gamma [F] = [\gamma F] = 0.$$

Thus, in physical terms the source condition may be stated as follows: Across any surface, the normal component of the impressed current must be continuous.

In Section 6 we shall introduce the so-called initial data condition :

$$(92) \qquad J^\mu \Big[u(0,x) \Big] = 0 \qquad \text{(for all } J^\mu\text{)}.$$

For Maxwell's equations this becomes

$$R^{\mu}\Big[u(0,x)\Big] = 0; \quad \mu = 1,2 \tag{93}$$

Equation (93) is equivalent to the two equations

$$[\gamma H] = 0 \quad \text{at} \quad t = 0; \tag{94}$$

$$[\gamma E] = 0 \quad \text{at} \quad t = 0. \tag{95}$$

Equation (94) is a direct consequence of (74). Equation (95) must be interpreted as an additional condition on the initial data of the problem. In physical terms, (95) states that there shall be no 'surface charges' present.

Summarizing the results of this section we may say that

1) Maxwell's equations are symmetric hyperbolic and satisfy the multiplicity condition.

2) The boundary condition for perfectly conducting boundaries is a special case of the type we considered in Section 1.

3) The source condition is satisfied for impressed currents which satisfy (91).

4) The initial data condition is a consequence of the divergence equations (73), (74) provided we exclude surface charges.

It follows from 1), 2) and 3) above that our formal expansion, and the entire contents of this paper are applicable to Maxwell's equations. The significance of 4) is discussed in Section 6.

3. The Duhamel integral theorem

In the introduction to Part II, we stated that the results of Part I play a key role in the proof of the asymptotic nature of our formal solution. In Part II we have been concerned with an inhomogeneous hyperbolic equation (2) (and its related reduced equation (4)) whereas in Part I we were concerned with the homogeneous equation $Lu = 0$. The connection between these two equations is established by the Duhamel integral theorem. Before proving this theorem, however, we must discuss the notion of 'weak solutions'.

As in Part I, we consider a space $\mathcal{S}_\Omega$ of smooth testing functions $\Omega(t,x)$ with compact support. More precisely $\mathcal{S}_\Omega$ will be the collection of all smooth k component vector functions Ω each of which vanishes identically outside some compact region R which depends upon Ω and lies in the half space $t > 0$. (For problems with boundaries we assume that R does not intersect the boundary.) For any u and f in C^1, the set of functions with continuous first derivatives, and any Ω in $\mathcal{S}_\Omega$ we have the identities

$$(96) \qquad \iint \Omega\, Lu\, dxdt = \iint u\, M\Omega\, dxdt; \quad \iint \Omega\, f_t\, dxdt = -\iint f\, \Omega_t\, dxdt.$$

Here the operator M, the adjoint of L is defined by

$$(97) \qquad M\Omega = -\sum_{\nu=0}^{n} (A^\nu \Omega)_{x_\nu} + B'\Omega .$$

B' denotes the transpose of the matrix B. The integration in (96) extends over the half space $t \geq 0$. The identities are obtained simply by integration by parts, the boundary terms vanishing because Ω has compact support. If now we have $Lu = f_t$, it follows trivially that

$$(98) \qquad \iint uM\Omega\, dxdt + \iint f\, \Omega_t\, dxdt = 0, \quad \text{for all } \Omega \text{ in } \mathcal{S}_\Omega.$$

Conversely if u and f are in C^1 in the neighborhood of some point in the half space $t > 0$ and satisfy (98) it follows that $Lu = f_t$ in that neighborhood. We may however consider more general classes of functions than C^1. Thus if u is any integrable function which satisfies (98) we shall say that u is a weak solution of the equation $Lu = f_t$. If, in addition, $u(0,x) = g(x)$ (and u is in T on $x_n = 0$)[*] we say that u is a weak solution (I) of the problem

$$(99) \qquad Lu = f_t; \quad u(0,x) = g(x), \qquad (u \text{ in } T \text{ on } x_n = 0).$$

We shall also need a second definition of weak solutions. For this purpose we consider a second space $\mathcal{S}_\Phi$ of smooth testing functions Φ, which have the same properties as the functions Ω except that we now require only that the function Φ vanishes identically outside some region R whose intersection with the half space $t \geq 0$ is compact (see Fig. 3). (For problems with boundaries we again assume that R does not intersect the boundary.) It is clear that S_Ω is a subset of S_ϕ. We now consider the analogue of (98)

$$(100) \qquad \iint u\, M\Phi\, dxdt + \iint f\, \Phi_t\, dxdt = 0 \qquad \text{for all } \Phi \text{ in } \mathcal{S}_\Phi .$$

As before if u and f are in C^1 in a neighborhood of any point in the half space $t > 0$, then $Lu = f_t$ in that neighborhood. If now, u and f are in C^1 in a neighborhood of a point $(0,x)$ on the initial plane $t = 0$, we may integrate by parts to obtain

---

[*] This is the boundary condition defined in Section 1. In the following sections we shall frequently treat initial value problems and initial boundary value problems simultaneously. Parentheses are used to denote the additional conditions or results associated with the problem with boundaries.

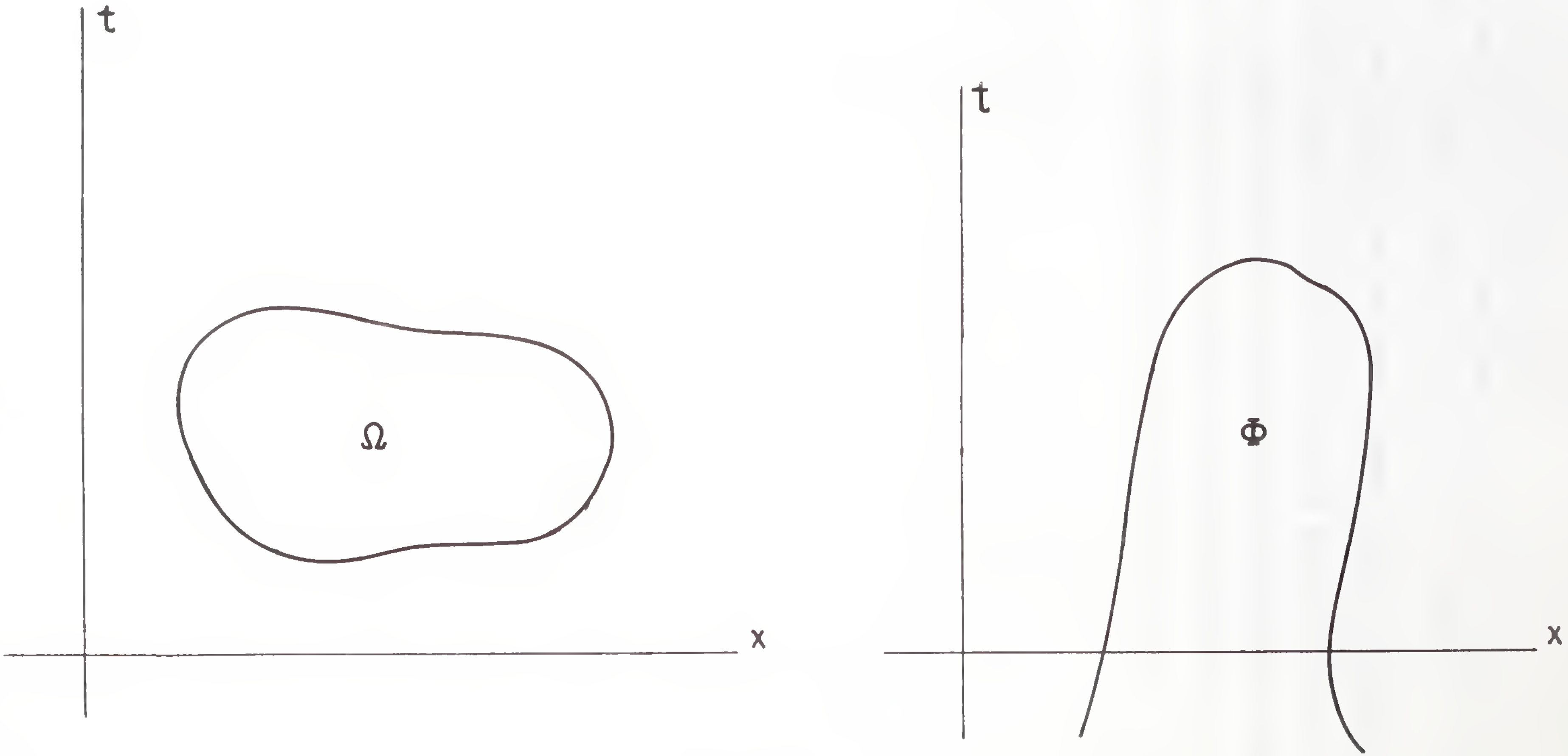

Figure 3

The support of typical testing functions Ω and Φ

$$(101)\quad 0 = \iint \Phi\,(Lu-f_t)\,dx\,dt + \int_{t=0} \Phi(0,x)\left\{u(0,x)-f(0,x)\right\}dx$$

$$= \int_{t=0} \Phi(0,x)\left\{u(0,x) - f(0,x)\right\} dx.$$

It follows that

$$(102)\qquad u(0,x) = f(0,x).$$

If u and f are any integrable functions which satisfy (100) and f_t exists (and u is in T on $x_n = 0$) we say that u is a weak solution (II) of the problem

$$(103)\qquad Lu = f_t; \quad u(0,x) = f(0,x); \quad (u \text{ in } T \text{ on } x_n = 0)^{*}.$$

In Part I we introduced the class $\mathcal{K}$ of piece-wise smooth functions: For each vector function u in $\mathcal{K}$, defined in the region of interest $\mathcal{D}$[**], there are a finite number of smooth hypersurfaces which divide $\mathcal{D}$ into subregions $\mathcal{D}_\nu$ such that the first derivatives of u can be made continuous in the closure of each $\mathcal{D}_\nu$ by appropriate definition of the derivative on the boundary.

Our two definitions of weak solutions are related by the following

> Lemma: If u and f are in $\mathcal{K}$, then u is a weak solution (I) of the problem (103) if and only if it is a weak solution (II) of the same problem.
>
> Proof:

1. Let u in $\mathcal{K}$ be a weak solution (II). Then, as before, $u(0,x) = f(0,x)$. Since S_{Ω} is a subset of S_{Φ}, (98) holds. Hence u is a weak solution (I).

2. Let u in $\mathcal{K}$ be a weak solution (I), and let C^{ν} denote the hypersurfaces subdividing $\mathcal{D}$ (for both functions, u and f). Then for any Φ in $\mathcal{S}_{\Phi}$

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[*] By setting $f = \int_0^t q(x,\tau)\,d\tau + \ell(x)$, we may define a weak solution (72) of the initial value problem $Lu = q(x,t)$; $u(0,x) = \ell(x)$.

[**] $\mathcal{D}$ is either the half space $t \geq 0$, or in problems with boundaries, that portion of the half space in which the solution is defined.

$$\iint uM\Phi\,dxdt + \iint f\,\Phi_t dxdt = \iint \Phi\left\{Lu-f_t\right\}dxdt + \int_{t=0}\Phi(x,0)\left\{u(0,x)-f(0,x)\right\}dx + \sum_{\nu}\int_{C_\nu}\left\{\Phi\,,\,u,f\right\} \tag{104}$$

where

$$\int_{C_\nu}\left\{\Phi\,,u,f\right\}$$

denotes an integral over C_ν involving the jumps in u and f. As we have seen before, $Lu = f_t$ except on the C_ν. Also $u(0,x) = f(0,x)$, hence the first two terms on the right side vanish. Thus

$$\iint u\,M\,\Phi\,dxdt + \iint f\,\Phi_t\,dxdt = \sum_{\nu}\int_{C_\nu}\left\{\Phi\,,u,f\right\}. \tag{105}$$

For any Ω in $\mathcal{S}_{\Omega}$

$$\iint u\,M\,\Omega\,dxdt + \iint f\,\Omega_t\,dxdt = \sum_{\nu}\int_{C_\nu}\left\{\Omega\,,\,u,\,f\right\}. \tag{106}$$

It is clear that we may choose an Ω which is equal to Φ on the hypersurfaces C_ν. Since the left side of (106) is zero, u must satisfy (100). Hence u is a weak solution (II).

We come now to the Duhamel integral theorem. The theorem was deduced by R. K. Luneburg, for Maxwell's equations, using Laplace transforms. M. Kline proved it for piecewise smooth solutions of Maxwell's equations and the second order linear hyperbolic equation by a process of direct substitution which is quite laborious. The following version is both more general and more direct:

Theorem: Let u be a weak solution (II) of the problem

$$Lu = 0; \quad u(0,x) = g(x); \qquad (u \text{ in } T \text{ on } x_n = 0). \tag{107}$$

Let $h(t)$ be a smooth scalar function of t and set

(108) $$W(t,x) = \frac{\partial}{\partial t}\int_0^t u(s,x)h(t-s)ds.$$

Then W is a weak solution (II) of the problem,

(109) $$LW = g(x)h'(t); \quad W(0,x) = g(x)h(0); \quad (W \text{ in } T \text{ on } x_n = 0).$$

Proof: If we consider the boundary condition as in (55) it is easy to see that W satisfies the condition if u does. Now by definition

(110) $$\iint uM\Phi\, dxdt + \iint g(x)\Phi_t\, dxdt = 0 \qquad \text{for all } \Phi \text{ in } \mathscr{S}_\Phi .$$

Using integration by parts we have

(111) $$\int_0^\infty WM\Phi\, dt = \int_0^\infty \frac{\partial}{\partial t}\left[\int_0^t u(s,x)h(t-s)ds\right] M\Phi(t,x)dt$$

$$= -\int_0^\infty \left[\int_0^t u(s,x)h(t-s)ds\right]\left[M\Phi(t,x)\right]_t dt$$

$$= -\int_0^\infty u(s,x)\int_s^\infty \left[M\Phi(t,x)\right]_t h(t-s)dtds$$

$$= -\int_0^\infty u(t,x)\int_t^\infty \left[M\Phi(s,x)\right]_t h(s-t)dsdt.$$

Let

(112) $$\Psi(t,x) = -\int_t^\infty \Phi_t(s,x)h(s-t)ds = -\int_0^\infty \Phi_t(z+t,x)h(z)dz.$$

It is easily seen that Ψ is in $\mathscr{S}_\Phi$. Also

(113) $$M\Psi = -\int_0^\infty \left[M\Phi(z+t,x)\right]_t h(z)dz = -\int_t^\infty \left[M\Phi(s,x)\right]_t h(s-t)ds.$$

(Compare the last terms of (113) and (111).) Now

$$\int_0^\infty \Psi_t(t,x)dt = -\int_0^\infty\int_0^\infty \Phi_{tt}(z+t,x)h(z)dzdt = -\int_0^\infty h(t)\int_0^\infty \Phi_{tt}(t+z,x)dzdt \tag{114}$$

$$= \int_0^\infty h(t)\,\Phi_t(t,x)dt.$$

From (111), (113), (110), (114),

$$\iint \mathrm{WM}\,\Phi\, dxdt = \iint u\,M\,\Psi\, dxdt = -\int g(x)\int \Psi_t\, dtdx \tag{115}$$

$$= -\iint g(x)h(t)\,\Phi_t\, dtdx,$$

and the proof is complete.

In what follows we shall deal only with functions u, w, f which are in $\mathcal{K}$. By the lemma, we may without ambiguity refer simply to 'weak solutions'. Setting $h(t) = -e^{-i\omega t}/i\omega$ we obtain the following

Corollary: Let u in $\mathcal{K}$ be a weak solution of the problem

$$Lu = 0; \quad u(0,x) = g(x); \qquad (u \text{ in } T \text{ on } x_n = 0) \tag{116}$$

Set

$$w(t,x) = \frac{\partial}{\partial t}\left[\frac{-e^{-i\omega t}}{i\omega}\int_0^t u(s,x)e^{i\omega s}\,ds\right] \tag{117}$$

$$= -\frac{u(t,x)}{i\omega} + e^{-i\omega t}\int_0^t u(s,x)e^{i\omega s}ds.$$

Then w is in $\mathcal{K}$ and is a weak solution of the problem

$$Lw = g(x)e^{-i\omega t}; \quad w(0,x) = -\frac{g(x)}{i\omega}; \qquad (w \text{ in } T \text{ on } x_n = 0). \tag{118}$$

In the remainder of this paper, u and w will denote the functions defined by (116) and (117),[*] where $g(x)$ is of type g. Theorem II-B of Part I asserts that under appropriate conditions, which will be satisfied, the solution u of (116) exists and is unique in $\mathcal{K}$. We have shown above that (117) is a solution of (118). The uniqueness of the solution of (118) follows easily from the uniqueness assertion of Theorem II-B. In order to avoid confusion it is useful to remember that the function $g(x)$ plays three roles: It is the initial data for u, a factor in the inhomogeneous term of the equation for w, and is the inhomogeneous term in the reduced equation, $Nv = g$.

4. The steady state function

We begin by imposing a new condition on the operator L which guarantees that the function $u(t,x)$ decays to zero as $t \to \infty$:

decay condition: If u is the solution of the problem

$$Lu = 0; \qquad u(0,x) = g(x); \qquad (u \text{ in } T \text{ on } x_n = 0) \tag{119}$$

where $g(x)$ is any function of type g, there is a constant $\omega_o > 0$ such that for every x in $\mathcal{D}$, and for $\omega \geq \omega_o$ the integral

$$\int_o^\infty u(t,x) e^{i\omega t} dt$$

exists and is bounded independently of ω.[**]

Since for any n, u_{t^n} is the solution of a problem of the type (119) it follows immediately from the above condition that

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[*] In the work of Luneburg and Kline, u is called the 'pulse solution'.

[**] For certain operators L it can be proved that the decay condition is satisfied. This is done in the Appendix.

$$\int_0^\infty u_{t^n}(t,x)e^{i\omega t}\,dt$$

exists and is bounded independently of ω for $\omega \geq \omega_n > 0$. Now, by integrating by parts it is easy to show that

$$u(T)e^{i\omega T} = i\omega\int_0^T ue^{i\omega t}\,dt + \int_0^T u_t\, e^{i\omega t}\,dt + c$$

where c is independent of T. Letting $T \to \infty$ we see that the right side of the above equation tends to a limit. Thus we may write

$$\lim_{t\to\infty} u(t,x)\, e^{i\omega t} = a\,.$$

But since the integral

$$\int_0^\infty u(t,x)e^{i\omega t}\,dt$$

converges it must be that $a = 0$. We conclude that $u(t,x) \to 0$ as $t \to \infty$. Applying the decay condition, (117) may be rewritten

(120) $$w(t,x) = z(t,x) + e^{-i\omega t}\,v(x)$$

where

(121) $$z(t,x) = \frac{-u(t,x)}{i\omega} - e^{-i\omega t}\int_t^\infty u(s,x)e^{i\omega s}\,ds$$

and

(122) $$v(x) = \int_0^\infty u(s,x)\, e^{i\omega s}\,ds.$$

It follows at once that $z(t,x) \to 0$ as $t \to \infty$. The function z is called the transient and v(x) is called the steady state function of the problem (118).

Note that v is uniquely determined by the decomposition (120) and the condition that $z \to 0$ as $t \to \infty$. Furthermore it is easily seen that v is independent of the initial data of the problem (118), for we may modify that problem by adding to w the weak solution u' of any problem

$$(123) \qquad Lu' = 0; \quad u'(0,x) = g'(x); \qquad (u' \text{ in } T \text{ on } x_n = 0)$$

where g' is of type g. The resulting function $w' = w+u'$ will be the weak solution of the problem

$$(124) \qquad Lw' = g(x)e^{-i\omega t}; \quad w'(0,x) = g'(x) - \frac{g(x)}{i\omega}; \quad (w' \text{ in } T \text{ on } x_n = 0).$$

But problem (124) has a steady state function which is again given by (122).

The steady state function, v, defined above is our main object of interest in this paper. We shall attempt to show that the formal expansion $\tilde{v}$ derived in Section 1 is an asymptotic expansion of v for $\omega \to \infty$. To this end we shall employ the integral expression (122). In Section 6 we shall show that v is a solution of the reduced equation, $Nv = g$.

5. Asymptotic expansion of the steady state function

In Part I we proved that under appropriate conditions the function $u(t,x)$ is piece-wise smooth, i.e., u is smooth everywhere in $\mathcal{D}$ except on certain hypersurfaces C^{α} where u and its derivatives have jump discontinuities. In Section 6 we will show that these hypersurfaces can never have a tangent plane parallel to the t-axis, so they can be defined by equations of the form $t = \Psi^{(\alpha)}(x)$. With this information, the asymptotic expansion of the integral (122) is obtained simply by repeated integrations by parts. At each step we obtain contributions from the lower end point of the integral and from the discontinuities of u and its derivatives. We shall assume that the discontinuities of u and its derivatives with respect to t all occur on a finite number of hypersurfaces $t = \Psi^{(\alpha)}(x)$; $\alpha = 1,\ldots,p$. We write

$$(125)\qquad v(x) = \int_{0}^{\Psi^{(1)}} + \int_{\Psi^{(1)}}^{\Psi^{(2)}} + \cdots + \int_{\Psi^{(p)}}^{\infty} u(t,x)e^{i\omega t}dt \,.$$

Integrating by parts

$$(126)\qquad v(x) = -\frac{u(0,x)}{i\omega} - \sum_{\alpha} \frac{e^{i\omega\Psi^{(\alpha)}}}{i\omega}\left[\, u \,\right]^{\alpha} - \left\{ \int + \cdots + \int u_t(t,x)\frac{e^{i\omega t}}{i\omega}\, dt \right\} .$$

Here

$$(127)\qquad \left[\, u \,\right]^{\alpha} = u\left(\Psi^{(\alpha)} + 0, x\right) - u\left(\Psi^{(\alpha)} - 0, x\right).$$

By repeated integrations by parts we obtain

$$(128)\qquad v(x) = \sum_{r=0}^{M} \left(\frac{-1}{i\omega}\right)^{r+1} \left\{ u_{t^r}(0,x) + \sum_{\alpha} e^{i\omega\Psi^{(\alpha)}(x)} \left[u_{t^r} \right]^{\alpha} \right\}$$

$$+ \left(\frac{-1}{i\omega}\right)^{M+1} \int_{0}^{\infty} u_{t^{(M+1)}}\, e^{i\omega t}\, dt \,.$$

Since u is a piecewise smooth solution of $Lu = u_t + \sum_{\nu=1}^{n} A^{\nu} u_{x_\nu} + Bu = 0$, and $u(0,x) = g(x)$, we have $u_t(0,x) = -\mathcal{L}g$, and in general

$$(129)\qquad u_{t^r}(0,x) = (-\mathcal{L})^r g.$$

The operator $\mathcal{L}$ is defined by (8).

Applying the decay condition, the integral in (128) may be estimated independently of ω. We substitute an order symbol for the last term in the sum together with the integral term:

$$v(x) = -\sum_{r=0}^{M-1} \frac{\mathcal{L}^r g}{(i\omega)^{r+1}} + \sum_{\alpha} e^{i\omega\Psi^{(\alpha)}(x)} \sum_{r=0}^{M-1} \left(\frac{-1}{i\omega}\right)^{r+1} \left[u_{t^r}\right]^{\alpha} + O\left(\frac{1}{\omega^{M+1}}\right) . \tag{130}$$

If g(x) is sufficiently smooth, we know from Part I that u will be smooth and the terms involving $[u_{t^r}]^\alpha$ will vanish. Now compare (130) with (10). We see that $\tilde{v} = y$ is indeed an asymptotic expansion to M terms of v. In the general case, where g and its derivatives are discontinuous across Γ, we must determine the functions $(-1)^{(r+1)}[u_{t^r}]^\alpha$ which appear in (130). We shall identify them with the terms $z^{(\alpha)(r+1)}$ in the formal expansion (see (34) and (13)). We must also identify the functions $\Psi^{(\alpha)}$ appearing above with the $\Psi^{(\alpha)}$ in (13).

Thus we must study the discontinuities of u. This is done in the next section.

6. The discontinuities of u

The present section is essentially a repetition of Sections 2, 3 and 6 of Part I. There is, however, one important difference: In Part I we studied discontinuities in derivatives of u normal to the discontinuity surfaces. Any other non-tangential direction of differentiation would do as well. In particular differentiation in the t-direction is permissible provided this direction is never tangential to a discontinuity surface*. In this section we shall use the t-direction. This, together with the fact that the coefficient matrices A^ν, B are independent of t, provides a considerable simplification. Furthermore, we shall need the results in terms of t-derivatives in order to determine the

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* Discontinuity surfaces (i.e., characteristic surfaces) may well have tangent planes parallel to the t-axis. This fact, which is true for Maxwell's equations (71) and (72), is commonly unrecognized. However such surfaces will be eliminated by imposing conditions on the initial data, g(x).

$[u_{t^r}]^{\alpha}$ of (130). The calculations will not be carried out fully, but only to the point where we may compare them with Section 1.

The function u is defined by the following:

$$\iint uM\Omega \, dxdt = 0 \qquad \text{for all } \Omega \text{ in } \mathcal{S}_{\Omega}, \tag{131}$$

$$u(0,x) = g(x) \tag{132}$$

$$u \text{ is in } \mathcal{K}, \tag{133}$$

and, for problems with boundaries,

$$(u \text{ in } T \text{ on } x_n = 0). \tag{134}$$

Suppose u is smooth on either side of the smooth hypersurface C, whose equation is $t = \Psi(x)$. We introduce the regular transformation

$$\phi = t - \Psi(x), \qquad y = x. \tag{135}$$

Here $y = y_1, \ldots, y_n$. We easily obtain the transformed operators

$$Lu = Gu_{\phi} + \sum_{\nu=1}^{n} A^{\nu} u_{y_{\nu}} + Bu \tag{136}$$

$$M\Omega = -(G\Omega)_{\phi} - \sum_{\nu=1}^{n} (A^{\nu}\Omega)_{y_{\nu}} + B'\Omega \tag{137}$$

where

$$G = I - \sum_{\nu=1}^{n} \Psi_{x_{\nu}} A^{\nu}. \tag{138}$$

If we integrate by parts in (131) and use the fact that Ω is arbitrary we obtain

$$G[u] = 0. \tag{139}$$

The symbol $\left[u\right] = u(\Psi+0,x) - u(\Psi-0,x)$ denotes the jump in u across C. Since A^ν and B are independent of ϕ, we may differentiate the equation $Lu = 0$ to obtain $Lu_{\phi^{(r-1)}} = 0$ on both sides of C. Hence

$$(140) \qquad G\left[u_{\phi^r}\right] + \sum_{\nu=1}^{n} A^\nu \left[u_{\phi^{(r-1)}}\right]_{y_\nu} + B\left[u_{\phi^{(r-1)}}\right] = 0; \qquad r = 1,2,\ldots \, .$$

Since $\left[u_{\phi^r}\right] \neq 0$ for some r, we must have

$$(141) \qquad \det G = 0.$$

It follows from (141) that the surface C is a characteristic surface for the operator L. Let $R^1,\ldots,R^k$ be a complete set of orthonormal eigenvectors of G. We now impose the multiplicity condition of Section 1. It follows that on C certain eigenvalues; $\lambda_1,\ldots,\lambda_q$ of G are identically zero and all the others are non-zero. Let $R^1,\ldots,R^q$ denote the associated null eigenvectors.

Now observe that $u_{\phi^r} = u_{t^r}$, and that the function $\left[u_{\phi^r}\right]$ is independent of ϕ. Set

$$(142) \qquad \left[u_{\phi^r}\right] = \left[u_{t^r}\right] = (-1)^{(r+1)} z^{(r+1)}(y) = (-1)^{(r+1)} z^{(r+1)}(x), \; r = 0,1,2\ldots$$

$$z^{(0)} = 0.$$

Then (139) and (140) may be written

$$(143) \qquad Gz^{(r+1)} - \mathcal{L} z^{(r)} = 0; \qquad r = 0,1,2,\ldots \, .$$

Now compare (143) with (15); (138) with (16); (141) with (17). It is clear that we may expand our $z^{(r)}$ as in (21) and obtain transport equations for the expansion coefficients which are identical to the transport equations of Section 1, namely (32) and (24)*. It follows that we may identify our functions Ψ and $z^{(r)}$ with those of Section 1 provided we can show that they are determined by the same initial conditions.

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* Some confusion might result from our definition of rays. The rays of Part II are the projections into the plane t = 0 of the rays of Part I.

The function u will be discontinuous across those characteristic surfaces $C^{(\alpha)}$ which intersect the initial plane $t = 0$ in the manifold Γ, across which g is discontinuous. Let the equation of $C^{(\alpha)}$ be $t = \Psi^{(\alpha)}(x)$. Then

$$\Psi^{(\alpha)} = 0 \quad \text{on } \Gamma. \tag{144}$$

Since $C^{(\alpha)}$ must be tangent to Γ, the projection into the plane $t = 0$ of the normal to $C^{(\alpha)}$ must be parallel to the normal to Γ. Thus

$$\Psi^{(\alpha)}_{x_\nu} = \rho(\alpha)\gamma_\nu \; ; \qquad \nu = 1,\ldots,n \tag{145}$$

Here $\gamma_1,\ldots,\gamma_n$ are components of the unit normal to Γ and $\rho(\alpha)$ is a proportionality factor which is determined by the equation

$$\det G^{(\alpha)} = \det\Bigl(I - \sum_{\nu=1}^{n} \Psi^{(\alpha)}_{x_\nu} A^\nu\Bigr) = \det(I - \rho(\alpha)E) = 0, \quad \text{on } \Gamma \tag{146}$$

where

$$E = \sum_{\nu=1}^{n} \gamma_\nu A^\nu. \tag{147}$$

There will be as many characteristic surfaces $C^{(\alpha)}$ emanating from Γ as there are distinct solutions $\rho(\alpha) = \frac{1}{\beta(\alpha)}$ of (146), hence as many as there are distinct non-zero eigenvalues $\beta(\alpha)$ of E. Precisely as in Section 1, (page 44) we introduce the eigenvectors B^j and J^ν of E. Instead of a 'source condition' it is more appropriate to define an initial data condition: $J^{\nu}[\,g\,] = 0$; $\nu = 1,\ldots,p$; on Γ.

(The conditions are, of course, identical*.)

By using the fact that Lu = 0, we may obtain the values of u_{t^r} on the initial plane, t = 0. Thus

$$u_{t^r}(0,x) = (-\mathcal{L})^r g(x); \quad r = 0,1,2,\ldots \quad . \tag{148}$$

The initial values for the transport equations are obtained from the fact that

$$\left[u_{t^r}(0,x)\right]^{\Gamma} = -\sum_{\alpha} e(\alpha)\left[u_{t^r}\right]^{\alpha}; \quad \text{on } \Gamma; \quad r = 0,1,2,\ldots \quad . \tag{149}$$

Here $[\;]^{\Gamma}$ and $[\;]^{\alpha}$ denote jumps across Γ and across the characteristic surface $C^{(\alpha)}$ and $e(\alpha) = \pm 1$. The orientation of $[\;]^{\alpha}$ is already defined. We choose the orientation of $[\;]^{\Gamma}$ to coincide with the orientation of $[\;]^{\Gamma}$ as defined in Section 1. By appropriately indexing the characteristic surfaces $C^{(\alpha)}$ to correspond to the waves $z^{(\alpha)}$ of Section 1 it is not difficult to see that the numbers $e(\alpha)$ will have the same values here as in Section 1. Using (148) and (142) we may rewrite (149) as

$$\left[\mathcal{L}^{(r-1)}g\right]^{\Gamma} = \sum_{\alpha} e(\alpha) z^{(\alpha)(r)}; \quad \text{on } \Gamma; \quad r = 1,2,\ldots \quad . \tag{150}$$

Now compare (144) with (37); (146) with (41); (150) with (39). It is clear that the functions $z^{(\alpha)(r)}$ and $Y^{(\alpha)}$ are determined by the same initial conditions as they are in Section 1. Hence the functions in the two sections are identical.

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*The initial data function g is identical with the source function g of Section 1. Hence the introduction of a 'new' condition on g would appear to be superfluous. However the results of this section are applicable to initial value problems in general, just as is the whole of Part I. From this point of view we really have a condition on initial data. It is this condition which makes an analysis in terms of t-derivatives possible. For Maxwell's equations the condition is particularly interesting for, as we have seen, it sheds new light on the 'divergence equations' (73),(74).

For problems with boundaries we must apply the boundary condition (134). Suppose a characteristic surface $C^{(\alpha)}$, emanating from the manifold Γ in the initial hyperplane intersects the boundary B ($x_n = 0$) in an (n-1)dimensional manifold $\triangle$. We now consider all the characteristic surfaces $C^{(p)}$ which intersect B in $\triangle$ and attempt to choose initial conditions for the transport equations on these surfaces in such a way that the jumps in u and its derivatives will be compatible with the boundary condition.

The analysis in Section 1 (beginning on page 47) is directly applicable here. We need only identify Yx-space with tx-space; 'phase surface' with 'characteristic surface', etc. Again we require that the shadow condition shall hold. The phase surfaces which are $\left\{\begin{matrix}\text{incoming}\\ \text{outgoing}\end{matrix}\right\}$ with respect to B must be identified with the characteristic surfaces on which t $\left\{\begin{matrix}\text{decreases}\\ \text{increases}\end{matrix}\right\}$ as we proceed along a ray from $\triangle$. For convenience we shall also apply the terms 'incoming' and 'outgoing' to the characteristic surfaces $C^{(p)}$. Thus one of the incoming characteristic surfaces $C^{(p)}$ is our $C^{(\alpha)}$. The rest of them may be assumed to carry no discontinuities*.

The boundary condition may be stated in the form

$$(151) \qquad b^\nu u = 0; \qquad \text{on } x_n = 0; \qquad \nu = 1,\dots,m.$$

The vectors b^ν are independent of t. Differentiating repeatedly with respect to t, we obtain

$$(152) \qquad b^\nu u_{t^r} = 0; \qquad \text{on } x_n = 0; \qquad \begin{matrix}\nu = 1,\dots,m\\ r = 0,1,2,\dots\end{matrix} \quad .$$

Now, in obvious notation,

$$(153) \qquad \sum_p b^\nu \left[u_{t^r}\right]^p = b^\nu \left[u_{t^r}\right]^\triangle = \left[b^\nu u_{t^r}\right]^\triangle = 0; \qquad \begin{matrix}\nu = 1,\dots,m\\ r = 0,1,2,\dots\end{matrix}$$

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* If, by coincidence, two of the $C^{(\alpha)}$ emanating from Γ intersect B in the same $\triangle$, we may still treat them separately.

Using the notation defined in (142), (153) may be written

$$(154) \qquad \sum_p b^\nu z^{(p)(r)} = 0; \qquad \text{on } B; \quad \begin{array}{l} \nu = 1,\dots,m \\ r = 0,1,2,\dots \end{array} .$$

We observe that (154) is identical with (63). It follows that the $z^{(p)(r)}$ of this section and the corresponding functions of Section 1 have the same initial values on B, hence are identical.

Having determined all the discontinuity surfaces[*] of u and all the discontinuities of u_{t^r} $(r = 0,1,2,\dots)$ on these surfaces, we renumber them with the index α. Now, using the notation of (142) we may rewrite (130) as

$$(155) \qquad v(x) = -\sum_{r=0}^{M-1} \frac{\mathcal{L}^r g}{(i\omega)^{r+1}} + \sum_\alpha e^{i\omega\Psi^{(\alpha)}(x)} \sum_{r=1}^{M} \frac{z^{(\alpha)(r)}(x)}{(i\omega)^r} + O\left(\frac{1}{\omega^{M+1}}\right) .$$

Comparing this equation with (70) we see that under appropriate conditions, the formal expansion $\tilde{v}$ is indeed an asymptotic expansion to M terms of the steady state function v. These conditions will be summarized in Section 7.

We are now in a position to show that the steady state function v is a solution of the reduced equation. By this we mean that for any point x, not on Γ, v is in C^1 in a neighborhood of x and satisfies the equation

$$(156) \qquad Nv = \sum_{\nu=1}^{n} A^\nu v_{x_\nu} + (B - i\omega)v = g.$$

To show this, we use the integral expression (122), and split up the integral at a typical discontinuity surface $t = \Psi(x)$:

$$(157) \qquad v(x) = \int_0^{\Psi(x)} + \int_{\Psi(x)}^{\infty} u(t,x)e^{i\omega t}dt.$$

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[*] We assume that they are finite in number.

Differentiating

$$(158) \qquad v_{x_\nu} = \int_0^\infty u_{x_\nu} e^{i\omega t} dt - e^{i\omega\psi} \psi_{x_\nu} [u] .$$

By integration by parts,

$$(159) \qquad \int_0^\infty u_t e^{i\omega t} dt = -e^{i\omega\psi} [u] - u(0,x) - i\omega \int_0^\infty u e^{i\omega t} dt .$$

Using (158) and (159) we have

$$(160) \qquad Nv = \int_0^\infty \left\{ \sum_{\nu=1}^n A^\nu u_{x_\nu} + Bu + u_t \right\} e^{i\omega t} dt + e^{i\omega\psi} \left\{ I - \sum_{\nu=1}^n \psi_{x_\nu} A^\nu \right\} [u] + u(0,x)$$

$$= \int_0^\infty \left\{ Lu \right\} e^{i\omega t} dt + e^{i\omega\psi} G[u] + g(x) .$$

But $Lu = 0$ for $t \neq \psi(x)$ and from (139) $G[u] = 0$. Hence

$$(161) \qquad Nv = g \qquad \text{for } x \text{ not on } \Gamma .$$

Since g is discontinuous on Γ, it is too much to expect that v should be continuously differentiable on Γ. However it is easy to see that *v is everywhere continuous*. This follows from the integral expression (122) and the fact that the discontinuity surfaces of u are never tangential to the t-axis[*].

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[*] This last assertion is true because we were able to carry out the construction of the discontinuities of u without using characteristic surfaces tangential to the t-axis. This was possible only because of the initial data condition. Had we used the formulation of Part I we would have discovered that the initial conditions for the transport equations on such surfaces were zero.

7. Summary

In this section we summarize the main results of Part II. Statements which refer to problems with boundaries are enclosed in parenthesis:

Given the problem

$$Lw = g(x)e^{-i\omega t}; \qquad w(0,x) = h(x); \qquad (w \text{ in } T \text{ on } x_n = 0)$$

where

1) L is a symmetric hyperbolic operator with coefficients that are smooth functions of x,

2) g(x) and h(x) are of type g

3) g(x) satisfies the source condition

4) the multiplicity condition is satisfied

5) the caustic condition is satisfied

6) (the shadow condition is satisfied)

Then there exists a weak solution w(t,x) defined in the region $t \geq 0$ which is unique in $\mathcal{K}$. If, in addition,

7) the decay condition is satisfied

then w has the decomposition

$$w(t,x) = z(t,x) + e^{-i\omega t} v(x)$$

where $z(t,x) \to 0$ as $t \to \infty$. The steady state function v(x) is uniquely determined by this decomposition and is independent of the choice of the function h(x). It is continuous, and except on the discontinuities of g(x) is a solution of the reduced equation

$$Nv = g.$$

The formal expansion $\tilde{v}$ derived in Section 1 is an asymptotic expansion to M terms (for $\omega \to \infty$) of v. M will be as large as we please if the coefficients of L, (the boundary condition), and, except on the manifold Γ, the function g(x) are sufficiently smooth.

Appendix

The Behavior of u for Large t

In Section 4 we introduced the decay condition. This condition, unlike the others, cannot be verified simply by examination of the operator L, or by a study of the geometry of the rays of the problem. Thus, without further investigation, we cannot even be sure that the class of operators which satisfy the decay condition is not empty. Unfortunately, the mathematical literature on the subject of the behavior for large t of solutions of hyperbolic equations appears to be practically non-existent. In this section, we shall attempt to examine the subject.

Our approach to the problem is by the well-known method of energy inequalities[*]. Unfortunately, we shall have to place severe restrictions on the coefficient matrices of the operator L in order to arrive at the desired result. It appears likely that these restrictions are not inherent to the problem itself, but merely reflect the inadequacy of the method. Perhaps this discussion will stimulate further interest in the question.

The energy method leads to an estimate of the L^2 norm of the solution u. Thus, if we introduce the function

$$(162) \qquad N(t) = \sqrt{\int u^2(t,x)dx} \quad ,$$

we can, under appropriate conditions, show that $N(t) \to 0$ as $t \to \infty$. Unfortunately, this does not suffice to guarantee pointwise convergence of u to zero as $t \to \infty$. However, convergence in the L^2 norm of u and certain of its derivatives is a sufficient condition for pointwise convergence of u. This is a consequence of the following

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[*] This method plays an important role in uniqueness and existence theorems for hyperbolic equations. See, e.g., [4], [5].

Lemma (Sobolev) [13]: Let $v(x)$ be a vector function of $x = x_1,\dots,x_n$. Let v and its derivatives up to order n be square integrable over the entire x-space. Then

$$|v(x_1,\dots,x_n)|^2 \le 2^n\Big(||v||^2 + ||v_{x_1}||^2 + \dots + ||v_{x_n}||^2 + ||v_{x_1x_2}||^2 + \dots + ||v_{x_1x_2\dots x_n}||^2\Big). \tag{163}$$

Here $||v||^2 = \int v^2 dx$; and $|v| = \max_j |v_j|$, where v_j denotes a component of v. The terms on the right side of the inequality include all derivatives of v up to order n, excluding those which involve differentiation with respect to any of the x_j more than once.

We shall write the equation for u in the form

$$Lu = u_t + A^\nu u_{x_\nu} + Bu = 0. \tag{164}$$

Here we employ the summation convention. In order to apply Sobolev's lemma, we must first enlarge the system (164) to include not only u, but also the appropriate derivatives of u. This is done, simply by differentiation of (164) with respect to the required x variables. By introducing vectors and matrices with many more components we may express the enlarged system in vector form. Thus we introduce the vectors

$$\tilde{u} = \begin{bmatrix} u \\ u_{x_1} \\ \cdot \\ \cdot \\ \cdot \\ u_{x_n} \\ u_{x_1x_2} \\ \vdots \\ u_{x_1x_2\dots x_n} \end{bmatrix} \qquad \tilde{g} = \begin{bmatrix} g \\ g_{x_1} \\ \cdot \\ \cdot \\ \cdot \\ g_{x_n} \\ g_{x_1x_2} \\ \vdots \\ g_{x_1x_2\dots x_n} \end{bmatrix}. \tag{165}$$

The enlarged system now takes the form

$$\tilde{L}\,\tilde{u} = \tilde{u}_t + \tilde{A}^{\nu}\tilde{u}_{x_\nu} + \tilde{B}\,\tilde{u} = 0.$$

It is tedious to calculate the matrix $\tilde{B}$ explicitly. However it is easy to see that

$$(166)\qquad \tilde{A}^{\nu} = \begin{bmatrix} A^{\nu} & 0 & \cdot & \cdot & 0 & 0 \\ 0 & A^{\nu} & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & A^{\nu} & 0 \\ 0 & 0 & \cdot & \cdot & 0 & A^{\nu} \end{bmatrix}$$

Thus the k x k submatrices appearing along the diagonal of $\tilde{A}^{\nu}$ are all identical to A^{ν}, and the other entries of $\tilde{A}^{\nu}$ are zero. For our purposes it is sufficient to observe that the top row of k x k submatrices of the matrix $\tilde{B}$ consists of the matrix B on the left followed by zero matrices. This too is easily seen. We now define a space $\tilde{\mathcal{J}}$ of testing function $\tilde{\Omega}(t,x)$. The space $\tilde{\mathcal{J}}$ is defined exactly as is the space $\mathcal{J}$ in Part I, except that the vectors $\tilde{\Omega}$ have as many components as does $\tilde{u}$. It is convenient to introduce a special type of element of $\tilde{\mathcal{J}}$. Let

$$(167)\qquad \hat{\Omega} = \begin{bmatrix} \Omega \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}.$$

Here Ω is any element of $\mathcal{J}$. As before, we introduce the adjoint operator

$$\tilde{M}\,\tilde{\Omega} = -(\tilde{A}^{\nu}\tilde{\Omega})_{x_\nu} + \tilde{B}'\,\tilde{\Omega}.$$

We say that $\tilde{u}$ is a weak solution of the initial-value problem

$$(168)\qquad \tilde{L}\,\tilde{u} = 0; \qquad \tilde{u}(0,x) = \tilde{g}(x)$$

if and only if

$$(169) \qquad \iint \tilde{u}\,\tilde{M}\,\tilde{\Omega}\,dxdt = 0 \quad \text{for all } \tilde{\Omega} \text{ in } \tilde{\mathcal{S}}; \quad \tilde{u}(0,x) = \tilde{g}(x).$$

We assume that $g(x)$ is of type g and vanishes identically outside a compact region, and that L satisfies the multiplicity condition. We also assume that the caustic condition is satisfied for the problem $Lu = 0$; $u(0,x) = g(x)$. It is then easy to show that $\tilde{g}(x)$ is of type g, and vanishes outside a compact region, that $\tilde{L}$ satisfies the multiplicity condition, and that the caustic condition is satisfied for the problem (168). It is obvious that $\tilde{L}$ is a symmetric hyperbolic operator. It follows from Theorem 2 of Part I that there exists a unique solution $\tilde{u}$ of (168) in the class $\mathcal{K}$ of piecewise smooth functions. $\tilde{u}$ is defined for all $t \geq 0$.

We observe from (166), (167) and our observations about $\tilde{B}$ that

$$(170) \qquad \tilde{u}\,\tilde{M}\,\hat{\Omega} = u\,M\,\Omega.$$

Here u denotes the first k components of $\tilde{u}$. Hence from (169),

$$\iint u\,M\,\Omega\,dxdt = 0 \quad \text{for all } \Omega \text{ in } \mathcal{S}.$$

Thus u is the solution of the problem

$$(171) \qquad Lu = 0; \quad u(0,x) = g(x);$$

which is considered in the definition of the decay condition.

Before deriving the energy inequality, we wish to recall the first steps in the calculation of the discontinuities of a solution of a discontinuous initial value problem[*]. By integration by parts,

$$(172) \qquad 0 = \iint \tilde{u}\,\tilde{M}\,\tilde{\Omega}\,dxdt = \iint \tilde{\Omega}\,\tilde{L}\,\tilde{u}\,dxdt + \int_C \tilde{\Omega}\left[\tilde{G}\,\tilde{u}\right]ds.$$

---

[*] This is mentioned in Part II, Section 6, but is carried out in detail in Part I, Section 2.

Here

(173) $$\tilde G = I - \Psi_{x_\nu} \tilde A^\nu$$

and $[f] = f(\Psi+0,x) - f(\Psi-0,x) = f_+ - f_-$ denotes the jump across the characteristic surface, C, whose equation is $t = \Psi(x)$. It follows that

(174) $$[\tilde G \tilde u] = \tilde G[\tilde u] = 0.$$

From (174) we observe that

(175) $$\left[\tilde u \tilde G \tilde u\right] = \tilde u_+ \tilde G \tilde u_+ - \tilde u_+ \tilde G \tilde u_- + \tilde u_- \tilde G \tilde u_+ - \tilde u_- \tilde G \tilde u_- = \tilde u_+ \tilde G[\tilde u] + \tilde u_- \tilde G[\tilde u] = 0.$$

Now, by integration by parts,

(176) $$0 = \int_0^T\!\!\int \tilde u \left\{\tilde L \tilde u\right\} dxdt = \int_0^T\!\!\int \tilde u \tilde M \tilde u \, dxdt - \sum_p \int_{C_p} \left[\tilde u \tilde G \tilde u\right] ds + \int \tilde u^2 \, dx \Big|_0^T . \; {}^*$$

The C_p denote the discontinuity surfaces of u. We introduce the function

(177) $$\tilde N^2(t) = \int \tilde u^2(t,x)dx$$

and rewrite the adjoint operator in the form

(178) $$\tilde M \tilde\Omega = -\tilde L \tilde\Omega - \tilde Q \tilde\Omega .$$

Here

(179) $$\tilde Q = \tilde A^\nu_{x_\nu} - \tilde B - \tilde B'.$$

Now from (176), using (175), (177), (178),

(180) $$0 = -\int_0^T\!\!\int \tilde u \tilde Q \tilde u \, dxdt + \tilde N^2(T) - \tilde N^2(0).$$

---

* Here the integrals exist because for each fixed t, $\tilde u(t,x)$ vanishes outside a compact region. This is a consequence of the same property of $\tilde g(x)$.

By differentiation with respect to T,

(181) $$2\tilde{N}\,\dot{\tilde{N}}(t) = \int \tilde{u}\,\tilde{Q}\,\tilde{u}\,dx.$$

We now impose the following restriction on the operator L:

energy condition: For every $\tilde{v}$, $\tilde{v}\,\tilde{Q}\,\tilde{v} \le -2\gamma\,\tilde{v}^2$; $\gamma > 0$; γ independent of x. Thus we require that the *energy matrix*, $\tilde{Q}$, be negative definite. Since this matrix can be calculated, the energy condition is an algebraic condition on the coefficient matrices of the operator L. For a given operator one can easily determine whether or not the energy condition is satisfied.

Now (181) may be rewritten

(182) $$\dot{\tilde{N}} \le -\gamma\,\tilde{N}.$$

It follows that

(183) $$\tilde{N}(t) \le C_1 e^{-\gamma\tau}.$$

Here

(184) $$C_1^2 = \tilde{N}^2(0) = \int \tilde{u}^2(0,x)dx = \int \tilde{g}^2(x)dx.$$

From Sobolev's lemma,

(185) $$|u(t,x)|^2 \le 2^n \int \tilde{u}^2(t,x)dx = 2^n\,\tilde{N}^2(t) \le 2^n C_1^2 e^{-2\gamma\tau}.$$

Thus $|u(t,x)|$ decays exponentially as $t \to \infty$ for every x. This certainly suffices to guarantee that u satisfies the decay condition.

References

[1] Courant, R., and Hilbert, D. - Methoden des Mathematischen Physic; Vol. II, Chapters V, VI, Springer, Berlin, 1937.

[2] Keller, J.B. - Geometrical acoustics I. The theory of weak shock waves; J. Appl. Phys., 25, 938-947, 1954.

[3] Courant, R. and Lax, P.D., - The propagation of discontinuities in wave motion; Proc. of the Nat'l Acad. of Sci., 42, No. 11, 872-876, Nov. 1956.

[4] Friedrichs, K.O., - Symmetric hyperbolic linear differential equations; Comm. Pure and Appl. Math., VII, No. 2, 345-393, May, 1954.

[5] Lax, P.D., - On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic equations; Comm. Pure and Appl. Math., VIII, No. 4, 615-633, Nov., 1955.

[6] Friedrichs, K.O., - Differential equations without regard to type; unpublished.

[7] Lax, P.D., - Mixed problems for symmetric hyperbolic systems; unpublished.

[8] Knopp, K., - Theory of Functions. Part II, Chapter 5, Dover, N.Y., 1947

[9] Luneburg, R.K., - The mathematical theory of optics; lecture notes, Brown University, 1944.
- Propagation of electromagnetic waves; lecture notes, New York University, 1948.

[10] Kline, M., - An asymptotic solution of Maxwell's equations; Comm. Pure and Appl. Math., 4, No. 2-3, 225-263, 1951
- Asymptotic solution of linear hyperbolic partial differential equations; J. Rational Mech. and Anal., 3, No. 3, 315-342, 1954
- Asymptotic solutions of Maxwell's equations involving fractional powers of the frequency; Comm. Pure Appl. Math., VIII, No. 4, 595-614, 1955.

[11] Keller, J.B.; Lewis, R.M., and Seckler, B.D. - Asymptotic solution of some diffraction problems; Comm. Pure and Appl. Math., IX, No. 2, 207-265, May, 1956.

[12] Kay, I. and Keller, J.B. - Asymptotic evaluation of the field at a caustic; J. Appl. Phys., 25, No. 7, 876-883, 1954.

[13] Sobolev, S.L. - On a theorem of functional analysis; Mathematicheskii sbornik, N.S. 4, 471-497, 1938.

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